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NUMBER THEORY

Tambs Lyche, R. Une propriété des fonctions symétriques des n premiers nombres impairs. Avh. Norske Vid. Akad. Oslo. I. 1944, no. 9, 29 pp. (1945). [MF 16440]

Let $e(x)$ denote the highest power of 2 which divides the positive integer x; let ${}_nS_k$ denote the elementary symmetric function of degree k of the numbers $1, 3, \cdots, 2n-1$; let ${}_nC_k$ denote the binomial coefficient. The author shows that $e({}_nS_k)=e({}_nC_k)+e(n)$ if $e(k)=0$ and $e({}_nS_k)=e({}_nC_k)$ if $e(k)>0$. His method is to prove first the special case $n=2^r$ and then proceed by induction. In a note added at the end of the paper, the author states that, in the special case $n=2^r$, the result follows from a theorem of Bauer [Hardy and Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1938, p. 102, theorem 130].

R. D. James (Vancouver, B. C.).

Bunicky, E. Kettenbruchentwicklungen der Quadratwurzeln aus den ganzen rationalpositiven inexakten Quadraten. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1942, 20 pp. (1942). (German. Czech summary) [MF 16115]

If A denotes a positive integer which is not a square, then $\sqrt{A}$ has a continued fraction of the form

$$(1) \qquad [a, \overline{a_1, a_2, \cdots, a_n, 2a}],$$

where $n \geqq 0$; if $n>0$ the sequence $a_1, a_2, \cdots, a_n$ is symmetric [Legendre]. On the other hand, any continued fraction (1) of this kind represents a number of the form $\sqrt{A}$ with rational $A>0$; but if $n>0$, A may be a fraction. The author's problem is to characterize those of the continued fractions (1) for which A is an integer. *J. F. Koksma.*

Klee, V. L., Jr. On the equation $\phi(x)=2m$. Amer. Math. Monthly 53, 327–328 (1946). [MF 16725]

A solution of (1) $\phi(x)=2m$, m odd, $m>1$, has the form p^α or $2p^\alpha$, p a prime $4s-1$ [Carmichael, Bull. Amer. Math. Soc. 13, 241–243 (1907)]. Using this the author proves that the number of solutions of (1) is twice the number of ways in which m can be expressed in the form $n(2n+1)^k$, $2n+1$ prime. From this it is inferred that (1) has 0, 2 or 4 solutions.

N. G. W. H. Beeger (Amsterdam).

Bose, R. C., Chowla, S., and Rao, C. R. Minimum functions in Galois fields. Proc. Nat. Acad. Sci. India. Sect. A. 15, 191–192 (1945). [MF 16894]

If $f(x)$ is a polynomial of degree n, with integral coefficients, which is a factor (mod p) of the polynomial whose roots are the primitive (p^n-1)th roots of unity, the authors call $f(x)$ a minimum function of $GF(p^n)$. The following theorem is proved. Let $f(x)=x^3+ax^2+bx+c$ be irreducible (mod p), where (i) a, b are not simultaneously congruent to 0 (mod p), (ii) $-c$ is a primitive root of p. Then $f(x)$ is a minimum function of $GF(p^3)$ if (iii) p^3+p+1 is a prime, or (iv) $(p^3+p+1)/3$ is a prime. It is stated that it is easy to generalize the theorem for $f(x)$ of degree n. *L. Carlitz.*

Bose, R. C., Chowla, S., and Rao, C. R. On the roots of a well known congruence. Proc. Nat. Acad. Sci. India. Sect. A. 15, 193 (1945). [MF 16938]

Making use of the $GF(p^3)$ the authors prove that the congruence

$$(C) \qquad x^3+x^2-2x-1 \equiv 0 \pmod{p}$$

is solvable when the prime $p \equiv \pm 1 \pmod 7$; this had been proved by Gauss using cyclotomy. [Note that the transformation $x=z+1/z$ takes (C) into $(z^7-1)/(z-1)=0$.] The roots of (C) are given by

$$\Theta^{m(p^3-1)/7}+\Theta^{-m(p^3-1)/7},$$

where $m=1, 2, 3$, and Θ is a primitive element of $GF(p^3)$; it is easily seen that these numbers are indeed in the $GF(p)$.

L. Carlitz (Durham, N. C.).

Vijayaraghavan, T., and Chowla, S. Short proofs of theorems of Bose and Singer. Proc. Nat. Acad. Sci. India. Sect. A. 15, 194 (1945). [MF 16939]

Singer [Trans. Amer. Math. Soc. 43, 377–385 (1938)] proved that, for $m=p^n$, p a prime, one can find $m+1$ integers $d_0, d_1, \cdots, d_m$, such that there exists exactly one pair of integers i, j satisfying $d_i-d_j \equiv n \pmod{m^2+m+1}$, provided $n \not\equiv 0 \pmod{m^2+m+1}$. Using the $GF(p^3)$ the authors give a direct proof of this theorem in the case $m=p$; they state that the proof is easily extended to the general case. It is also stated that Bose's affine analogue of Singer's theorem [J. Indian Math. Soc. (N.S.) 6, 1–15 (1942); these Rev. 4, 33] is proved similarly. *L. Carlitz* (Durham, N. C.).

Potts, D. H. Solution of a Diophantine system proposed by Bhaskara. Bull. Calcutta Math. Soc. 38, 21–24 (1946). [MF 16969]

Bhaskara's problem of finding integers x, y such that $x-y$ is a square and x^3+y^3 a cube is essentially the problem of solving the Diophantine system $x-y=t^2$, $x^2+y^2=z^3$. The author considers, separately, two general solutions $x=r^3-3rs^2$, $y=3r^2s-s^3$ and $x=r(r^2+s^2)$, $y=s(r^2+s^2)$ of the second equation and imposes on each the condition $x-y=t^2$. With the aid of results of Kronecker, Desboves, and Legendre [Dickson's History of the Theory of Numbers, vol. 2, Washington, 1920, pp. 169, 405, 570], he derives formulas which may be used to obtain an infinite number of numerical solutions. Several numerical solutions, including those of Bhaskara, are given. *W. H. Gage.*

Duarte, F. J. On the equation $x_1^3+x_2^3=y_1^3+y_2^3$. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 7, 855–866 (1943). (Spanish)

The solution of this equation in integers is reduced to the problem of representing a certain integer by a quadratic form of the type v^2-Au^2. A solution equivalent to the well-known formulas of Euler and Binet is then obtained and it is shown that this furnishes all integral solutions of the equation. *H. W. Brinkmann* (Swarthmore, Pa.).

Duarte, F. J. On the equation $x^3+y^3+z^3=0$. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **8**, 971–979 (1944). (Spanish)

A simple proof is given of the impossibility of solving the Fermat equation in the title in nonzero integers. It is based on writing the equation in the form

$$(x+y+z)^3=(x+y)(y+z)(z+x)$$

and then showing that it is possible to set $x=3t+u$, $y=6t-u$. The proof avoids the usual discussion of the divisibility of x, y, z by 2 and 3. A solution of the equation in terms of numbers in certain quadratic fields is also given.

H. W. Brinkmann (Swarthmore, Pa.).

Bussi, C. Osservazione sull'ultimo teorema di Fermat. Boll. Un. Mat. Ital. (2) **5**, 42–43 (1943). [MF 16086]

If $a^n+b^n=c^n$, $(a, b)=(b, c)=(c, a)=1$, then (I) $a-A$, $b-A$, $c-A$, A arbitrary, have no common factor; (II) from $\varphi(a^n+b^n)\equiv 0 \pmod n$, n prime, $n>2$, follows $\varphi(a^n)\equiv\varphi(b^n)\equiv\varphi(c^n)\equiv 0 \pmod n$ and hence a, b, c have, respectively, prime factors p, q, r for which $p\equiv q\equiv r\equiv 1$ or $p=n$, $q\equiv r\equiv 1 \pmod n$. Elementary proofs. *N. G. W. H. Beeger.*

Podsypanin, V. On the equation $ax^4+bx^2y^2-cy^4=1$. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 105–114 (1946). (Russian. English summary) [MF 16678]

The author proves that the equation of the title, where a, b, c are integers, a and c positive, has at most one solution in positive integers. The proof is based on a study of special units in a field of order 8. The result generalizes one of B. N. Delaunay and Faddeev. [See Skolem, Diophantische Gleichungen, Ergebnisse der Math., v. 5, no. 4, Springer, Berlin, 1938, pp. 113 f.; B. N. Delone, Izvestiya Rossiĭskoĭ Akad. Nauk [Bull. Acad. Sci. Russie] (6) **16**, 253–272 (1922); D. K. Faddeev, Učenye Zapiski Leningrad. Gos. Pedagog. Inst. **28**, 141–145 (1938); Delone and Fadeev, Theory of irrationalities of third degree, Acad. Sci. URSS. Trav. Inst. Math. Stekloff, v. 11, 1940, pp. 289–313; these Rev. **2**, 349.] *K. Mahler* (Manchester).

Banerjee, D. P. On the rational solutions of the Diophantine equation $ax^n-by^n=k$. Proc. Benares Math. Soc. (N.S.) **5**, 29–30 (1943). [MF 14438]

S. Chowla [Indian Phys.-Math. J. **5**, 5–6 (1934)] has proved the following theorem. If Euler's conjecture is true, that $x_1^r+\cdots+x_m^r\neq y^r$ where $1<m<r$ and $r>2$, then the Diophantine equation $ax^n-by^n=k$, $n>2$, has at most one rational solution if n is odd and $|a|+|b|<n/2$. The author proves some simple corollaries of this result.

A. Brauer (Chapel Hill, N. C.).

Cotlar, M., and Levi, B. Exercises on the cosine function. Math. Notae **5**, 193–214 (1945). (Spanish) [MF 16372]

The authors consider the distribution of the values of sequences of the form $\{\cos(2\pi n_k x+2\pi\alpha_k)\}$ or equivalently of sequences of fractional parts of $n_k x+\alpha_k$. Using measure-theoretic arguments they show that, if $\{n_k\}$ is strictly increasing, the sequence is, for almost all x, dense in $(0, 1)$. [Stronger results are known; cf., for example, Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, pp. 8, 94.] The authors also discuss the frequency of the n_k for which

$$|\cos 2\pi n_k x-\cos 2\pi n_{k_s} x|$$

exceeds a given positive number. *R. P. Boas, Jr.*

Davenport, H., and Mahler, K. Simultaneous Diophantine approximation. Duke Math. J. **13**, 105–111 (1946). [MF 15879]

Theorem 1(a). If $c>2/\sqrt{23}$ and α, β are any two irrational numbers, then there exist an infinity of fractions p/r, q/r for which $(\alpha-p/r)^2+(\beta-q/r)^2<c/r^3$. (b) This is false if $c<2/\sqrt{23}$. Theorem 2(a). If $c>2/\sqrt{23}$ and α, β are any two real numbers for which 1, α, β are linearly independent, then there exist an infinity of sets of integers p, q, r, satisfying $|\alpha p+\beta q+r|<c/(p^2+q^2)$, $p^2+q^2>0$. (b) This is false if $c<2/\sqrt{23}$.

This result is interesting because the problem of finding the best possible value of $c>0$ for which the simultaneous inequalities

$$|\alpha-p/r|<c/r^3, \quad |\beta-q/r|<c/r^3$$

have an infinity of rational solutions p/r, q/r $(r>0)$ is still unsolved. The proof of (a) in both cases is based on investigations on star bodies by Mahler [in course of publication] and on a theorem of Davenport on three dimensional lattices [Proc. London Math. Soc. (2) **45**, 98–125 (1939)]. The proof of (b) in each case is based on the properties of the cubic field $k(\phi)$, where ϕ is the real root of $t^3-t-1=0$.

J. F. Koksma (Amsterdam).

Mordell, L. J. Further contribution to the geometry of numbers for non-convex regions. Trans. Amer. Math. Soc. **59**, 189–215 (1946). [MF 15650]

Minkowski's theory on lattice points in convex regions led him to the following problem. If $f(x, y)\leqq 1$ denotes a convex, symmetrical domain with its center at the origin, to find a parallelogram of minimum area, one of whose vertices is at the origin and whose other three vertices are on the boundary $f(x, y)=1$. The author extends this problem to a quite general class of nonconvex infinite domains $f(x, y)\leqq 1$, very similar to the special region R: $|x^n+y^n|\leqq 1$ (more precisely written: $|\operatorname{sgn} x|x|^n+\operatorname{sgn} y|y|^n|\leqq 1$, where n denotes a real positive number), develops a method for solving the problem and discusses the application to the special region R in full detail for $n\geqq 3$. *J. F. Koksma.*

Ollerenshaw, Kathleen. Lattice points in a circular quadrilateral bounded by the arcs of four circles. Quart. J. Math., Oxford Ser. **17**, 93–98 (1946). [MF 16775]

Those arcs of the four circles $x^2+y^2=\pm\lambda x$ and $x^2+y^2=\pm\mu y$ which do not pass through the origin form the boundary of a nonconvex star domain K. The author determines $\Delta(K)$ and the critical lattices of K by means of Mordell's method [Proc. London Math. Soc. (2) **48**, 339–390 (1945); these Rev. **6**, 257]. *K. Mahler* (Manchester).

Monna, A. F. Généralisation P-adique d'un théorème de Minkowski sur les formes linéaires. Nederl. Akad. Wetensch., Proc. **49**, 162–166 = Indagationes Math. **8**, 59–63 (1946). [MF 16569]

In the space R_n of all points $x=(x_1, \cdots, x_n)$ with P-adic coordinates, let $\|x\|=\max_{1\leqq i\leqq n}|x_i|_P$ be the distance of x from the origin. Denote by

$$y_i=\sum_{j=1}^{n} a_{ij}x_j, \qquad i=1, 2, \cdots, n,$$

a system of n linear forms with P-adic coefficients of non-vanishing determinant Δ and by

$$x_i=\Delta^{-1}\sum_{j=1}^{n} A_{ij}y_j, \qquad i=1, 2, \cdots, n,$$

the inverse forms. Assume that S is a set of points x in R_n without finite points of accumulation; let $\varphi(\lambda)$ be the number of its elements such that

$$\|x\| \leqq P^{\lambda}$$

and let $\limsup_{\lambda\to\infty} P^{n\lambda}/\varphi(\lambda) = a$. The author proves that, if

$$m(I_0) = |\Delta|_P^{-n} \prod_{i=1}^{n} \max_{1\leqq j\leqq n} P^{q_j} |A_{ij}|_P > a,$$

then there exist two different elements a, b of S such that $x = a-b \neq 0$ satisfies the inequalities

$$\text{(P)} \qquad \left|\sum_{j=1}^{n} a_{ij}x_j\right|_P \leqq P^{q_i}, \qquad j = 1, 2, \cdots, n.$$

Here $m(I_0)$ is the P-adic measure of the parallelepiped defined by (P). *K. Mahler* (Manchester).

Lehmer, D. H. Two nonexistence theorems on partitions. Bull. Amer. Math. Soc. 52, 538–544 (1946). [MF 16814]

Let d be 0 or a positive integer and let $q_d(n)$ be the number of partitions of n into parts differing by d or more. The author proves:

$$\text{(1)} \qquad \sum_{n=0}^{\infty} q_d(n)x^n = \sum_{s=0}^{\infty} \frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)};$$

(2) corresponding to any $d>2$, there is no set S of positive integers such that, for every n, $q_d(n)$ is equal to the number of partitions of n into parts taken from S; (3) corresponding to any $d\neq 1$, there is no set S of positive integers such that, for every n, $q_d(n)$ is equal to the number of partitions of n into distinct parts taken from S. *T. Estermann.*

Linnik, U. V. On the possibility of a unique method in certain problems of "additive" and "distributive" prime number theory. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 3–7 (1945). [MF 16396]

The author outlines a new proof of Vinogradov's theorem on sums of three primes, which will have a more homogeneous structure than the original proof. The "trigonometrical sieve" method of Vinogradov is avoided by using Linnik's deep results on the distribution of the zeros of the L-functions. Some of these results are consequences of earlier papers [see the same C. R. (N.S.) 41, 145–146 (1943); Rec. Math. [Mat. Sbornik] N.S. 15(57), 3–12, 139–178, 347–368 (1944); these Rev. 6, 58, 260]; others are contained in a paper in course of publication.

H. Davenport (London).

Turán, P. Über die Wurzeln der Dirichletschen L-Funktionen. Acta Univ. Szeged. Sect. Sci. Math. 10, 188–201 (1943). [MF 16746]

The author proves that the total number $N_k(\alpha)$ of zeros of all the L-functions (mod k) in the rectangle $\alpha\leqq\sigma\leqq 2$, $|t|\leqq 5$ satisfies

$$N_k(\alpha) = O(k^{4(1-\alpha)} \log^{11/2} k)$$

for $\sqrt{2}-\frac{1}{2}\leqq\alpha\leqq 1-2(\log k)^{-1}$ and satisfies

$$N_k(\alpha) = O(k^{\frac{1}{2}+2\alpha-2\alpha^2} \log k)$$

for $\frac{1}{2}+5(\log\log k)^{-1}\leqq\alpha\leqq\sqrt{2}-\frac{1}{2}$. It follows that "almost all" the L-functions (mod k) have no zeros in the region $\sigma\geqq\frac{1}{2}+5(\log\log k)^{-1}$, $|t|\leqq 5$. An important part in the proof is played by estimates for $\sum_\chi |f(s,\chi)-1|^2$, where

$$f(s,\chi) = L(s,\chi) \sum_{n\leqq x} \chi(n)\mu(n)n^{-s},$$

with a modification when χ is the principal character, and x is a certain function of k. *H. Davenport* (London).

Marengoni, A. Sulla scomponibilità di un intero nella somma di due quadrati. Period. Mat. (4) 22, 182–185 (1942).

Stöhr, Alfred. Anzahlabschätzung einer bekannten Basis h-ter Ordnung. Math. Z. 47, 778–787 (1942). [MF 15914]

The basis of order h considered is the following one defined by D. Raikov [Rec. Math. [Mat. Sbornik] N.S. 2(44), 595–597 (1937)]. Denote by A_q the set of all numbers of the form $\sum_{i=0}^{\infty} 2^{q-1}z_i 2^{hi}$, $z_i = 0, 1$. Put $A = \{A_1, A_2, \cdots, A_h\}$. It is known that A is a basis of order h for the set of all integers and that

$$A(x)+1 < x^{1/h} \max_{1\leqq t\leqq h} [(h+t)2^{-(t-1)/h}].$$

The author refines this inequality and obtains

$$A(x)+h-1 < x^{1/h} \max_{1\leqq t\leqq h} [(h+t)2^{-(t-1)/h}(1-2^{-h})^{1/h}] = x^{1/h}K.$$

He shows further that $A(x)+h > x^{1/h}K$ for infinitely many x. From these two results it follows that $\lim_{x\to\infty} A(x)x^{-1/h} = K$.

H. B. Mann (Columbus, Ohio).

Erdös, Paul, and Niven, Ivan. The $\alpha+\beta$ hypothesis and related problems. Amer. Math. Monthly 53, 314–317 (1946).

Expository article.

Erdös, P. Some remarks about additive and multiplicative functions. Bull. Amer. Math. Soc. 52, 527–537 (1946). [MF 16813]

It is known that the distribution function $f_1(x)$ of the multiplicative number-theoretic function $n/\phi(n)$ is purely singular and has as its spectrum the half-axis $1\leqq x\leqq\infty$. The author obtains sharp asymptotic inequalities for the distribution function $f_1(x)$ at the ends of the spectrum $x=1$ and $x=\infty$. It is also shown that, if $g(n)$ is any function such that $\log\log\log n = o(g(n))$, $n\to\infty$, then

$$\sum_{m=n}^{n+g(n)} \phi(m)/m = (1+o(1))6g(n)/\pi^2$$

and that the number of integers m between n and $n+g(n)$ for which $m/\phi(m)\leqq x$ is equal to $(1+o(1))g(n)f_1(x)$. These results do not hold if $\log\log\log n \neq o(g(n))$. Similar theorems are stated concerning the distribution of the function $\sigma(n)/n$. The methods of proof are direct counting processes depending on elementary estimates.

The author states the following theorem. Let $f(n)$ be an additive function possessing an asymptotic distribution function $F(x)$. Let $|f(p^k)|\leqq C$. Then, for any $c>0$, $1-F(x) < \exp(-cx)$ for sufficiently large x. However, it seems to the reviewer that in the proof [cf. the fifth formula line on p. 536] the assumption $f(p^k)\geqq 0$ is implicitly used. The proof of this theorem is made to depend on the following result. Let $g(n)\geqq 0$ be a multiplicative function. Then $g(n)$ possesses an asymptotic distribution function if and only if the two series $\sum_p (g(p)-1)'/p$, $\sum_p ((g(p)-1)')^2/p$ converge, where $(g(p)-1)'$ denotes 1 or $g(p)-1$ according as $|g(p)-1|$ does or does not exceed 1. The proof of this theorem is reduced to the case of additive functions [Erdös and Wintner, Amer. J. Math. 61, 713–721 (1939); these Rev. 1, 40] by considering $\log g(n)$. This, however, supposes that $g(n)$ is positive, rather than nonnegative.

The paper also contains a discussion of the distribution of the numbers $b=b_r(n)$ defined by the equation $(p_r^{a_r})^b=p_{r+1}^{a_{r+1}}$, where $n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, $p_1^{a_1}<p_2^{a_2}<\cdots<p_k^{a_k}$.

P. Hartman (Baltimore, Md.).

Wintner, Aurel. The values of the norms in algebraic number fields. Amer. J. Math. **68**, 223–229 (1946). [MF 16418]

A nonartificial example is given of a multiplicative function $F(n)$, whose mean value $M(F)=\lim_{m\to\infty}\sum_1^m F(n)/m$ exists, but which is not almost periodic (B). Let $\mathfrak{R}$ be an algebraic number field of degree greater than 1 and let $F(n)$ denote the number of integral ideals of $\mathfrak{R}$ whose norm is n. It is proved that $F(n)=0$ for almost all natural numbers n. Since, on the other hand, $M(F)\neq 0$ ($M(F)$ being the residue at $s=1$ of Dedekind's ζ-function $\zeta(s,\mathfrak{R})$), $F(n)$ cannot be almost-periodic (B). [The assertion (12) is derived from the erroneous formula (11): $F(p^i)\leqq C$, C independent of i and p. However, (12) can easily be proved otherwise.]

N. G. de Bruijn (Eindhoven).

Benneton, Gaston. Sur l'arithmétique des quaternions et des biquaternions. Ann. Sci. École Norm. Sup. (3) **60**, 173–214 (1943). [MF 14641]

In part I, a survey is given of the arithmetic of integral quaternions (with integer coordinates). The methods are based on residues and multiples mod m and the uniqueness and existence of factors of primitive quaternions, rather than on the use of a g.c.d. In part II [cf. C. R. Acad. Sci. Paris **216**, 262–264 (1943); these Rev. **6**, 38] biquaternions (Cayley octaves) are taken as quaternion pairs $A=[A_1, A_2]$, with the multiplication formula

$$[A_1, A_2][B_1, B_2]=[A_1B_1-\bar{B}_2A_2,\ B_2A_1+A_2\bar{B}_1].$$

The conjugate of A is $\bar{A}=[\bar{A}_1, -A_2]$ and the norm $N(A)=A\bar{A}$ is the sum of the squares of the eight coordinates. Multiplication is in general neither associative nor commutative, but $(AB)\bar{B}=A(B\bar{B})$. The author considers integral biquaternions, with all eight coordinates rational integers. If m is an odd positive integer, the class mod m of A consists of all biquaternions B such that $B\equiv uA \pmod m$, with u prime to m; if m is even there are adjoined to the class all biquaternions $uA+\frac{1}{2}mK \pmod m$, where K has all its coordinates unity. All biquaternions of a class mod m have the same right (left) divisors of norm m. Every class mod m contains a biquaternion of norm at most m^2; if m is odd, at most $8m^2/9$ (in ordinary quaternions, at most $4m^2/9$). Biquaternions of prime norm p and certain biquaternions of norm 4 are indecomposable. A process similar to one used in part I is developed for constructing factorizations $A=QQ_1$ with $N(Q)$ a preassigned norm dividing $N(A)$, and hence for factorization into indecomposable elements. Every biquaternion is a left divisor of an essentially unique reduced biquaternion of the form $1+x_1i_1$, $1+x_2i_2+x_3i_3$ or $1+x_4i_4+x_5i_5+x_6i_6+x_7i_7$. Hence the number of biquaternions of given norm is found, in agreement with classic results on the number of representations as a sum of eight squares. It is stated that, with 305 exceptions, all positive integers are sums of eight unequal positive squares. *G. Pall.*

Benneton, Gaston. Arithmétique des quaternions. Bull. Soc. Math. France **71**, 78–111 (1943). [MF 13234]

The results of part I of the article reviewed above are extended to give results on quaternions with the same left or right divisors and on orthogonal matrices of orders 3 and 4. These were, in part, previously announced [C. R. Acad. Sci. Paris **214**, 406–408 (1942); these Rev. **4**, 240]. The corresponding results for Hurwitz integral quaternions are also given. The results are more general in some respects than similar ones of the reviewer [Trans. Amer. Math. Soc. **47**, 487–500 (1940); these Rev. **2**, 36]. There seems to be an error in the author's remarks [p. 94] on the representations of a quaternion as a sum of squares. *G. Pall.*

ALGEBRA

Kaplansky, Irving, and Riordan, John. The problem of the rooks and its applications. Duke Math. J. **13**, 259–268 (1946).

In how many ways can k rooks be placed on a chessboard of given shape so that no two can attack each other? For an m by n rectangle the answer is $\binom{m}{k}\binom{n}{k}k!$. The authors obtain the answer for some other shapes; for example, for a right-angled isosceles triangle of side $n-1$ it is the Stirling number of the second kind, $S(n-k, n)=\Delta^{n-k}0^n/(n-k)!$. Various applications (for example, the classification of permutations by ascending runs) are described. The number of ways of putting j bishops on the black squares of an ordinary chessboard is found to be the coefficient of x^j in the symbolic expansion of $T^n(T+x)^k$, where

$$T^{n-1}=\sum_k S(n-k, n)x^k.$$

H. S. M. Coxeter (Toronto, Ont.).

Conte, Luigi. A proposito del metodo di Eulero per la risoluzione dell'equazione biquadratica. Period. Mat. (4) **23**, 65–72 (1943).

Amodeo, F. Sulla risoluzione della equazione di 4° grado. Period. Mat. (4) **24**, 3–9 (1946).

Vigil, Luis. Observations on a theorem of Rey Pastor on Graeffe's method. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 191–193 (1946). (Spanish) [MF 16922]

Let $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$. The theorem in question is that each root of $f(x)=0$ differs from a root of $f(x)=\delta$ by less than $|\delta|^{1/n}$. Ostrowski [Acta Math. **72**, 99–155, 157–257 (1940); these Rev. **1**, 323; **2**, 342] has pointed out that this is not necessarily a one-to-one correspondence between the roots of the two equations. The author gives another example and shows that the correspondence is one-to-one if δ is small enough. *R. P. Boas, Jr.* (Providence, R. I.).

Duncan, W. J. Factorization of a class of determinants and applications to dynamical chains. Philos. Mag. (7) **36**, 615–622 (1945). [MF 16978]

Let u be a square matrix of order n with distinct characteristic roots, so that there exists a nonsingular matrix k such that $kuk^{-1}=\rho$, where $\rho=[\rho_1, \rho_2, \cdots, \rho_n]$ is a diagonal matrix. Let P and C be square matrices of order m and let I_n be the unit matrix of order n and $F=P\cdot I_n+C\cdot u$, where $C\cdot u$ is the direct product of C and u. Then

$$I_m k(F)I_m\cdot k^{-1}=P\cdot I_n+C\cdot\rho,$$

so that $|F|=\prod_{s=1}^n|P+\rho_sC|$. Thus the determinant $|F|$ of order mn is factored into the product of n determinants of order m. This factorization theorem is applicable to the

determinants of motion of a large class of dynamical chains. The author works out in detail an application to the theory of segmented aircraft wings. *J. Williamson.*

Perelmann, M. Sur une propriété des suites des polynômes. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. **12**], 87–91 (1941). (Russian. French summary) [MF 16490]

The theorem proved is the following: if $\{f_s(x_1, x_2, \cdots, x_k)\}$ is an infinite sequence of polynomials in k variables, then there exists an integer n_0 such that every common zero of the first n_0 polynomials of the sequence is a zero of every polynomial of the sequence. In an added note the author explains that it was pointed out to him immediately before publication of the paper that his result is an immediate consequence of Hilbert's theorem of the basis in the theory of ideals. *O. Zariski* (Urbana, Ill.).

Abstract Algebra

Kořínek, Vladimír. Der Schreiersche Satz und das Zassenhaussche Verfahren in Verbänden. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1941**, 29 pp. (1941). [MF 16119]

Let L be a lattice with an undefined relation aNb, defined for some pairs a, b with $a \geqq b$; this is to be thought of as an abstract analogue of "b is a normal subgroup of a." Properties of the relation N are described, necessary and sufficient in order that the Schreier-Zassenhaus refinements of two "normal chains" be normal. Necessary and sufficient conditions are also given for corresponding quotients to be "similar" in the sense of Ore. In all, ten conditions on N are discussed in their axiomatic relationship to various parts of the Schreier-Zassenhaus theorem. [Cf. A. Uzkow, Rec. Math. [Mat. Sbornik] N.S. **4**(**46**), 31–43 (1938); R. J. Duffin and R. S. Pate, Duke Math. J. **10**, 743–750 (1943); these Rev. **5**, 170.] *G. Birkhoff* (Cambridge, Mass.).

Ribeiro, Hugo. What is a quadriculate? Gaz. Mat., Lisboa **7**, no. 27, 3–5 (1946). (Portuguese) [MF 15941]

A "quadriculate" is a finite rectangle of squares. It is proved that a lattice diagram is a quadriculate if and only if (i) every element covers at most two elements, (ii) the lattice is distributive, (iii) there are just four elements with complements. *G. Birkhoff* (Cambridge, Mass.).

Marinescu, Gh. Structures et nombres caracteristiques. Bull. Math. Soc. Roumaine Sci. **46**, 113–119 (1944). [MF 16514]

The author discusses various partially ordered sets as generalized "numbers." He notes that the "numbers" denoting the Boolean algebras of "closed" subsets of T_0-spaces include all order-types as subnumbers. The general point of view is that of the reviewer [Duke Math. J. **9**, 283–302 (1942); these Rev. **4**, 74]. *G. Birkhoff.*

Byrne, Lee. Two brief formulations of Boolean algebra. Bull. Amer. Math. Soc. **52**, 269–272 (1946). [MF 16190]

It is shown that a Boolean algebra may be characterized as a system of not less than two elements closed under a binary operation $\times$ (juxtaposition) and a unary operation $'$, satisfying (I) $xy' = zz' \rightleftarrows xy = x$, (II) $(xy)z = x(yz)$ and (III) $xy = yx$. The operation $'$ may be realized as the logical complement and $\times$ either as logical sum or logical product. The author asserts that (I) to (III) are independent. *A. L. Foster* (Berkeley, Calif.).

Sperling, M. Rings, every subring of which is an ideal. Rec. Math. [Mat. Sbornik] N.S. **17**(**59**), 371–384 (1945). (Russian. English summary) [MF 16673]

Enumération de tous les anneaux possèdant un élément-générateur et dont chaque sous-anneau est un idéal. Voir le résumé, tout en corrigeant une erreur typographique: (1) $a_1^2 = 0$ au lieu de (1) $a_1 = 0$. *H. Freudenthal.*

Forsythe, Alexandra, and McCoy, Neal H. On the commutativity of certain rings. Bull. Amer. Math. Soc. **52**, 523–526 (1946). [MF 16812]

It was shown by Jacobson [Ann. of Math. (2) **46**, 695–707 (1945); these Rev. **7**, 238] that, if every element a of a ring R satisfies an equation $a^{n(a)} = a$, $n(a) > 1$, then R is commutative. The proof made use of general results on algebraic algebras. Noting that the proof can be made very short when R is a division ring, the present authors obtain a quite simple reduction of the general case to this case by means of the following main result. A regular ring is isomorphic to a subdirect sum of division rings if and only if it has no nilpotent elements. A corollary is that an algebraic algebra without nilpotent elements is a subdirect sum of division algebras. A completely elementary proof of Jacobson's theorem is given for the special case where there is a prime p such that $pa = 0$, $a^p = a$ for every $a \in R$. *I. S. Cohen* (Philadelphia, Pa.).

Cohen, I. S. On the structure and ideal theory of complete local rings. Trans. Amer. Math. Soc. **59**, 54–106 (1946). [MF 15318]

This paper provides a very thorough insight into the structure of a local ring (l.r.), a concept introduced by W. Krull, and contains proofs of certain conjectures he made [J. Reine Angew. Math. **179**, 204–226 (1938)]. A considerable contribution to their study was made by C. Chevalley [Ann. of Math. (2) **44**, 690–708 (1943); these Rev. **5**, 171]. A l.r. is a commutative ring in which every ideal has a finite basis and in which the non-units form an ideal $\mathfrak{m}$. It follows that $\mathfrak{m}$ is the unique maximal ideal of the ring and that the intersection of all its powers is the zero ideal. Part of the present investigation is based on a study of the properties of a generalised local ring (g.l.r.). A g.l.r. is a commutative ring with an identity element in which all the non-units form an ideal $\mathfrak{m}$ with a finite basis which has the zero ideal as the intersection of all its powers. Clearly a l.r. is a g.l.r. Several characterisations of l.r.'s among g.l.r.'s are given but it is pointed out that it is not yet known whether there exists a g.l.r. which is not a l.r.

A topology can be introduced in a l.r. by taking as neighbourhoods of the zero the powers of $\mathfrak{m}$. A complete g.l.r. is always a l.r. and in fact the completion of any g.l.r. is a l.r., which shows how easily a l.r. can be obtained from a g.l.r. The g.l.r.'s are only introduced because they turn up in the proofs of certain results concerning l.r.'s.

Krull's conjectures actually concern complete regular l.r.'s. For the definition of regularity (this term was introduced by Chevalley) it is necessary to introduce a basis $u_1, \cdots, u_n$ of the ring which is such that no proper subset of these elements forms a basis. (It is, however, shown that the choice of the basis is immaterial.) A l.r. is regular if for every form ϕ in n variables and of degree k, whose coeffi-

cients are in the ring but not all in $\mathfrak{m}$, we have

$$\phi(u_1, \cdots, u_n) \not\equiv 0 \ (\mathfrak{m}^{k+1}).$$

The author gives an equivalent definition and actually proves structure theorems for l.r.'s which are not necessarily regular. In order to discuss these theorems we define the residue field of a l.r. $\mathfrak{R}$ as the field P of residue classes of $\mathfrak{R}$ modulo $\mathfrak{m}$. The class of all l.r.'s can be subdivided into four categories: (1) when $\mathfrak{R}$ and P both have the characteristic zero; (2) when $\mathfrak{R}$ and P both have characteristic p (a prime number); (3) when $\mathfrak{R}$ has characteristic zero and P has characteristic p; (4) when $\mathfrak{R}$ has characteristic p^k ($k>1$) and P has characteristic p. A l.r. and its completion always belong to the same category. Examples are given of l.r.'s which belong to each of the four categories and it is finally shown that any complete l.r. is homomorphic to one of these examples. The main result is that every complete l.r. is homomorphic to a complete regular l.r., namely, the ring of all power series in a certain number of variables with coefficients from a field or from a valuation ring of a certain simple type. The proofs in the cases when $\mathfrak{R}$ and P have the same or different characteristics are essentially different. The main difficulty is the imbedding of the coefficient ring or field in $\mathfrak{R}$. Some of the tools used are similar to those employed by H. Hasse and F. K. Schmidt [J. Reine Angew. Math. 170, 4–63 (1933)], O. Teichmüller [J. Reine Angew. Math. 176, 141–152 (1936)] and S. MacLane [Ann. of Math. (2) 40, 423–442 (1939)] in connection with fields complete with respect to a valuation.

A section is devoted to regular l.r.'s and the structure of complete regular l.r.'s is described. Apart from this, questions concerning uniqueness of factorisation are dealt with and known results for rings of power series over infinite fields are extended to the case of finite coefficient fields. In addition, work by F. S. Macaulay [The Algebraic Theory of Modular Systems, Cambridge University Press, 1916] on polynomial rings is extended. *O. Todd-Taussky.*

Kiokemeister, Fred. The Asano postulates for the integral domains of a linear algebra. Bull. Amer. Math. Soc. 52, 490–495 (1946). [MF 16808]

Let g be an integral domain having a Noether ideal theory and A an algebra over its quotient field. If A is separable, it is known to possess a set of g-integers satisfying the Asano postulates [N. Jacobson, The Theory of Rings, Math. Surveys, v. 2, American Mathematical Society, New York, 1943, pp. 123–126; these Rev. 5, 31]. The author shows that it suffices to assume that A has a unity element.

I. Kaplansky (Chicago, Ill.).

Inoue, Hirosi. Eine Eigenschaft der Norm. Tôhoku Math. J. 49, 60–68 (1942). [MF 14696]

If k is a field, L an extension field of k, and K an extension field of L such that K has finite degree over k, then the norm satisfies the equation $N_{Kk}(\alpha)=N_{Lk}(N_{KL}(\alpha))$ for an arbitrary element α of K. This rule is often proved only under the assumption that K possesses a primitive element. The paper contains two proofs which do not depend on the existence of a primitive element. One of these proofs is due to T. Nakayama. Another proof for the same fact can be found in H. Weyl, Algebraic Theory of Numbers, Ann. of Math. Studies, no. 1, Princeton University Press, 1940 [these Rev. 2, 37]. *R. Brauer* (Toronto, Ont.).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions finies et séparables des corps valués complets: principes fondamentaux; espaces de polynomes et transformation T; lois d'unicité, d'ordination et d'existence. C. R. Acad. Sci. Paris 222, 626–628 (1946). [MF 16040]

Krasner, Marc. Théorie non-abélienne des corps de classes pour les extensions finies et séparables des corps valués complets: conducteur, théorie de l'irrégularité. C. R. Acad. Sci. Paris 222, 984–986 (1946). [MF 16388]

In these two notes the author's previous results [same C. R. 219, 345–347, 433–435, 473–476, 539–541 (1944); 220, 28–30, 761–763 (1945); 221, 737–739 (1945); 222, 37–40, 165–167, 363–365, 581–583 (1946); these Rev. 7, 363, 364, 508] are applied to develop a non-Abelian class field theory for finite separable extensions of complete valuation fields. Let k be a complete field with valuation, $\mathfrak{k}$ its algebraic closure, α an element of $\mathfrak{k}$ and $K=k(\alpha)$. Let $|\cdots|$ and $\omega(\cdots)$ denote valuation and order in $\mathfrak{k}$. Let $f_{\alpha/k}(x)$ be the minimal polynomial of α in k, let C_α be the largest circle with centre α which contains no other conjugate of α and let $C_{\alpha,q}$ be the circle $|x-\alpha|<r_q^{(\alpha)}=\exp(-\nu_q^{(\alpha)})$, where $\nu_q^{(\alpha)}$ is the qth characteristic number and $K_q^{(\alpha)}$ the qth characteristic field of K/k at α. The following three fundamental principles are proved. (I) If α is separable over k, then $\beta \varepsilon C_\alpha$ implies $k(\beta) \geq k(\alpha)$. (II) If, moreover, $[k(\beta):k] = [k(\alpha):k]$, then $k(\beta)=k(\alpha)$. (III) If $\beta \varepsilon C_{\alpha,q-1}$ then $k(\beta) \geq K_q^{(\alpha)}$.

The laws of uniqueness, ordering and existence result from consideration of polynomial spaces $E_{k,\nu}$ and $E_{K/k}^{\alpha}$ which consist, respectively, of all normalized irreducible polynomials in k of degree ν and all polynomials $f_{\beta/k}(x)$, where $\beta \varepsilon \mathfrak{k}$ and $k(\beta) \geq K$. The distance function or "ultrametric" in these polynomial spaces is defined by $d_a(f,g)=|R(f,g)^{a/mn}|$, where n and u are the degrees of f and g, $R(f,g)$ is their resultant and a is any real number.

There follows the theory of the conductor of K/k and a "theory of irregularity" for the case of non-Galois extensions. To each element of the Galois hypergroup $G_{K/k}$ is assigned an "irregularity number." These numbers measure, in a sense, the departure of K/k from the case of a Galois extension and give rise to a series of subhypergroups of $G_{K/k}$ analogous to the hypergroups of ramification. [The definition of the irregularity number [line 14, p. 985] is apparently misstated. It appears to the reviewer that the intention is to define it as the upper bound (on the semi-real axis) of $\omega(\sigma\alpha-\beta)$.] *D. C. Murdoch.*

THEORY OF GROUPS

***Borůvka, Otakar. Úvod do Teorie Grup. [Introduction to the Theory of Groups].** Královska Česká Společnost Nauk, Praha, 1944. 80 pp. (Czech)

This book presents an elementary introduction to the theory of abstract groups. An original feature is the relatively large space devoted to the discussion of groupoids. The purpose is to see more clearly which theorems on groups are independent of the assumptions of associativity and possibility of division. The book consists of three chapters of about equal length. The first gives the fundamental concepts of set theory, the second contains the theory of groupoids and the last one contains the theory of groups.

C. Loewner (Syracuse, N. Y.).

Mueller, Edith. The study of ornaments as application of the theory of groups of finite order. Euclides, Madrid **6**, no. 59, 42–52 (1946). (Spanish)

Rédei, L. Über den Fundamentalsatz der Abelschen Gruppen von endlicher Ordnung. Acta Univ. Szeged. Sect. Sci. Math. **10**, 109–111 (1941). [MF 15838]

The author presents an alternate proof of the basis theorem for Abelian groups by an induction based on a maximal subgroup instead of a factor group.

J. S. Frame (East Lansing, Mich.).

Götlind, Erik. Some theorems on the number of groups of order pq^r. Norsk Mat. Tidsskr. **28**, 13–16 (1946). (Swedish) [MF 16691]

The principal results are that there are at least as many non-Abelian abstract groups of order pq^r as there are of order q^r and that, when $p>q^r$ and $p\not\equiv 1 \pmod q$, the number of abstract groups of order pq^r is equal to the number of order q^r. *D. E. Rutherford* (St. Andrews).

Schmidt, O. Infinite soluble groups. Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 145–162 (1945). (Russian. English summary) [MF 16666]

The set M of normal subgroups of the group Γ is called soluble if (I) it contains (1) and Γ, (II) given two elements of M one of them contains the other, (III) the factor-group of two elements Γ_1, Γ_2 of M is Abelian whenever there is no element Γ_3 of M between Γ_1 and Γ_2. A group Γ is called soluble if ϕ/ψ possesses a soluble set whenever ϕ is any subgroup of Γ and ψ any normal subgroup of ϕ. The notions are treated in connection with minimality conditions and conditions of well-ordering and are applied to some special types of groups, for example, the locally soluble and locally finite groups [cf. S. Tchernikov, same Rec. N.S. **13(55)**, 317–333 (1943); these Rev. **6**, 201]. *H. Freudenthal.*

Yukiyosi Kawada. Ueber den Dualitätssatz der Charaktere nichtkommutativer Gruppen. Proc. Phys.-Math. Soc. Japan (3) **24**, 97–109 (1942). [MF 15021]

The duality relations between commutative groups and their character groups [see Pontrjagin, Topological Groups, Princeton Math. Ser., v. 2, Princeton University Press, 1940; these Rev. 1, 44] have been generalised to the non-commutative case by T. Tannaka [Tôhoku Math. J. **45**, 1–12 (1938)]. A different approach is made by the present author. For the case of finite groups he uses C-algebras, which are commutative algebras with respect to the complex numbers, where the multiplication constants $c_{\alpha\beta}^{\gamma}$ of the base elements x_i, $i=1, \cdots, n$, are real. In addition a C-algebra has a unit element $e=x_1$ and it possesses an involutory ring automorphism $u\leftrightarrow u'$ which is given by a permutation $\alpha\leftrightarrow\alpha'$ of the base elements. It is assumed that $c_{\alpha\beta}^{1}=\delta_{\alpha}^{\beta}h_\alpha$, $h_\alpha>0$, and that the correspondence $x_\alpha\rightarrow h_\alpha$ establishes a representation of the algebra. It follows that such an algebra has no radical and that it has n linearly independent representations $x_\alpha^{(\lambda)}$ by permutations. Consider the solutions $f^{(\lambda)}$ of the equations $\sum_{\alpha=1}^{n} f^{(\lambda)}x_\alpha^{(\lambda)}=\delta_\alpha^1 g$, where $g=\sum h_\alpha$. The characters are obtained by putting $\chi_\alpha^{(\lambda)}=h_\alpha^{-1}\sqrt{(f^{(\lambda)})}x_\alpha^{(\lambda)}$. This gives a shorter approach to the theorems on characters given by Hoheisel [Monatsh. Math. Phys. **48**, 448–456 (1939); these Rev. **1**, 104]. It is shown that the centre of the group ring of a finite group is a C-algebra A if the sums of the elements in a whole class are taken as base elements. An algebra based on the characters is introduced and it is proved that it is a C-algebra of the same rank as A. It is called the conjugate C-algebra $\bar{A}$ and it is finally shown that $A\cong\bar{A}$. Further correspondences between subalgebras of A and $\bar{A}$ are investigated.

The author next considers duality relations for a bicompact group G. A correspondence between the elements x of G and the sets $\bar{x}=\{axa^{-1}\}$, $a\in G$, is introduced. Let m denote the Haar measure in G and $\bar{m}$ the measure induced in $\bar{G}$. Functions $f(\bar{x})$ which are $\bar{m}$-integrable are then considered; they form a ring L_0 if we define the product of f and g by

$$[f*g](x)=\int_G f(xy^{-1})g(y)m(dy).$$

A norm is introduced by putting $\|f\|=\int_{\bar{G}}|f(\bar{x})|\bar{m}(d\bar{x})$. It is shown that L_0 can be made into a normed ring [see I. Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1940); these Rev. **3**, 51] and a one-to-one correspondence is set up between the maximal ideals of L_0 and the characters of G. The author then introduces the commutative ring $\bar{L}$ which is formed by the linear combinations of a finite number of $z_\alpha=\chi_\alpha/\chi_\alpha(1)$, where χ_α is a character of G. Using results by M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 275–280, 355–359 (1940); these Rev. **2**, 316] and G. Šilov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 83–84 (1940); these Rev. **2**, 314], he shows that the space $\bar{G}$ is determined uniquely by $\bar{L}$ and that the algebraic structure of L_0 is uniquely determined by $\bar{L}$ and C, where C is the Banach space of all continuous functions $f(x)$ on G with the norm $\|f\|=\max_{\bar{x}} f(\bar{x})$. *O. Todd-Taussky.*

Dietzman, A. P. On the multigroups of complete conjugate sets of elements of a group. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 315–317 (1946).

An ultragroup Z is defined as a multigroup which satisfies two additional axioms, namely, (I) there exists an element e such that for every a we have $ea\supset a$ and $ae\supset a$; (II) there is a finite sequence of submultigroups Z_i such that $Z\supset Z_1\supset Z_2\supset\cdots\supset Z_m=e$ and such that Z_{i-1}/Z_i is a group. A special partition of a group G is considered, where $a\sim b$ if $b=x^{-1}ax$ for some x. This partition leads to a multigroup Z, the subject of the principal theorem. This is that Z is an ultragroup if and only if G is solvable. Two auxiliary theorems are proved, although they are special cases of known results [see Campaigne, Amer. J. Math. **62**, 599–612 (1940), theorems 7.3, 7.1; these Rev. **2**, 7].

H. Campaigne (Arlington, Va.).

ANALYSIS

Aumann, Georg, und Haupt, Otto. Bemerkung über Abhängigkeit von Funktionen. Math. Z. **50**, 144–154 (1944). [MF 15844]

The theorem of K. Knopp and R. Schmidt [Math. Z. **25**, 373–381 (1926)] is proved under slightly weaker hypotheses. Given the equations $x_i'=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, with the f_i continuous in an open set D of (x)-space and with the f_i differentiable (in the sense of Stolz) except on a subset S of D having image of Lebesgue measure 0 in (x')-space, let C be a closed compact subset of D. If the Jacobian of the transformation is zero almost everywhere in $C-S$, then $f_1, \cdots, f_n$ are dependent in C. A partial converse is proved. Additional results are given.

A. B. Brown (Flushing, N. Y.).

Ghika, Al. Sur la nature fonctionnelle des fonctions quasi analytiques générales. Mathematica, Timişoara **20**, 148–165 (1944); **21**, 19–44 (1945). [MF 16381]

Une classe d'opérations fonctionnelles $x(p)$ définie sur un ensemble E est dite par l'auteur quasi-analytique sur E, relativement à une famille F de vrais sous-ensembles e de E, qui couvre E, si la coincidence de deux opérations de la classe sur un ensemble $e \varepsilon F$ (ensemble déterminant) entraine la coincidence sur E. L'auteur se réfère à des exemples: (1) les fonctions continues dans un domaine D, déterminées par leurs valeurs sur un dénombrable partout dense dans D; (2) les fonctions étudiées par D. Pompeiu, de la forme $f(x)=\int\int_E \varphi(z)(z-x)^{-1}d\omega$, où E est un ensemble borné, parfait, discontinu, de mesure superficielle positive, et $\varphi(z)$ est mesurable, bornée sur E; (3) les fonctions analytiques réelles de p variables: la condition nécessaire et suffisante pour qu'une suite de points $M_n \to 0$ dans l'espace euclidien à p dimensions E^p soit un ensemble e est qu'elle ait en 0 au moins un élément d'accumulation à $p-1$ dimensions; (4) les fonctions analytiques d'une variable complexe; la valeur $f(P)$ en un point $P \subset D$ est une fonctionnelle additive, de classe 1 de Baire au plus par rapport aux valeurs prises en une suite de points $M_n \to 0$, $0 \subset D$; (5) les fonctions holomorphes d'une variable hypercomplexe dans D à p $(2 \leqq p \leqq 4)$ dimensions: les suites $M_n \to 0$, $0 \subset D$, ayant en 0 un élément d'accumulation à $p-2$ dimensions sont des ensembles e; (6) les transformations périphèriques $P=f(p)$ de E^p dans E^q, c'est à dire telles que pour tout point $K \varepsilon E^q$ et tout domaine fermé $\delta+\varphi \varepsilon E^p$, le maximum de la distance KP pour $P \varepsilon \Delta+\Phi=f(\delta+\varphi)$ ait lieu pour p appartenant à la frontière φ de δ; l'auteur définit, pour terminer, certaines classes de telles transformations, plus générales que les fonctions holomorphes, qui sont des fonctionnelles linéaires des valeurs prises sur la frontière d'un domaine.

P. Lelong (Lille).

Theory of Sets, Theory of Functions of Real Variables

Eyraud, Henri. Schémas bifurqués et représentations transfinies. Ann. Univ. Lyon. Sect. A. (3) **3**, 25–32 (1940).

Consider all possible orderings of the integers, including, therefore, representatives of all denumerable order types. There is a continuum of representatives of each of a continuum of order types. A representation of each ordering is given by a bifurcated scheme, which is an infinite root-tree (in the sense of Cayley) placed in the plane, with appropriately numbered nodes. Special order types can be identified by properties of these schemes and each scheme is uniquely defined by a suitably chosen real number.

J. W. Tukey (Princeton, N. J.).

Eyraud, Henri. D'une représentation des ensembles fermés. Ann. Univ. Lyon. Sect. A. (3) **3**, 33–37 (1940).

A representation of closed sets in $(0, 1]$ by means of subsequences of the sequence of integers. The representation is based on infinite binary expansions of numbers.

J. W. Tukey (Princeton, N. J.).

Eyraud, Henri. Le problème du continu. Ann. Univ. Lyon. Sect. A. (3) **6**, 33–45 (1943).

Appealing implicitly to the axiom of choice or to the maximum principle (often called Zorn's lemma), the author observes that the transfinite ordinals of the second number class can be represented by a selected family of arithmetic functions, that is, functions from integers to integers. The ordering relation is $f(\omega)>g(\omega)$, where ω represents a place into which any integer can be substituted, if, and only if, $f(x)>g(x)$ for $x \geqq n$. The family selected is not known to be unique. Transfinite arithmetic beyond the first ϵ-number is briefly discussed. Each such arithmetic function is uniquely represented by a polygonal line, whose corners lie at the lattice points $(x, f(x))$, and which is called an f-line. A polygonal segment which passes through the lattice points $M_1, \cdots, M_n$ and which is part of a nondenumerable infinity of f-lines can always be extended in an infinite number of ways to a segment $(M_1, \cdots, M_n, M_{n+1})$ with the same properties. Using the incorrect assumption [corrected by the author; cf. the following reviews] that, if every segment of $(M_1, \cdots, M_i, \cdots)$ has this property, it follows that $(M_1, \cdots, M_i, \cdots)$ is an f-line, the author attempts to prove that every arithmetic function is majorized, in the sense described above, by one of the selected functions.

J. W. Tukey (Princeton, N. J.).

Eyraud, Henri. Les transfinis ordinaux des seconde et troisième classe. Ann. Univ. Lyon. Sect. A. (3) **7**, 5–13 (1944).

By considering a selected set of functions from and to ordinals of the first and second number classes, the author represents the transfinite ordinals of the third class. The results about polygonal lines given in the two adjacent reviews can be extended to this case. *J. W. Tukey.*

Eyraud, Henri. Le problème de la saturation. Ann. Univ. Lyon. Sect. A. (3) **8**, 47–48 (1945).

A polygonal line is an envelope line if its corners are at lattice points such that every initial segment is common to a nondenumerable infinity of f-lines [cf. the second preceding review]. Every f-line corresponding to an ordinal of the second class exceeding a fixed ordinal depending on the representation is an envelope line. Not every f-line is an envelope line. *J. W. Tukey* (Princeton, N. J.).

Shukla, P. D. On the differentiability of step functions. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 271–274 (1943).

A. N. Singh proposed the following question to the author: does the existence of $\Phi'(0)$ for a step function $\Phi(x)$ depend upon the existence of the metric density at $x=0$ of the set of values of x corresponding to the lines of invariability of $\Phi(x)$? Here step function means a continuous function having infinitely many lines of invariability in the neighborhood of $x=0$. The author answers this question in the negative, showing by two examples that the existence of the metric density at $x=0$ is neither necessary nor sufficient for the existence of $\Phi'(0)$. *A. Rosenthal.*

Shukla, P. D. On the differentiability of an indefinite integral. Bull. Calcutta Math. Soc. **38**, 25–33 (1946). [MF 16970]

Various conditions for the differentiability of monotonic functions at a point have been given by the author [same Bull. **37**, 9–14 (1945); these Rev. **7**, 9] in terms of a sequential variable. These are now restated as conditions that $F(x)=\int_0^x f(x)dx$ is differentiable at $x=0$. Examples are given to show that the sufficient conditions are not necessary.

P. Civin (Eugene, Ore.).

González Quijano, Pedro M. Derivative and continuity. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 121–131 (1946). (Spanish) [MF 16915]

The author makes some general remarks on processes for constructing functions with prescribed singular behavior. He constructs a particular continuous function by defining it first on the binary rationals and shows that it has no derivative at such points. He states that it has no derivative at any point. *R. P. Boas, Jr.* (Providence, R. I.).

Gomes, A. Pereira. *R* integrability of continuous functions. Gaz. Mat., Lisboa **7**, no. 28, 9–12 (1946). (Portuguese)

Ridder, J. Zur Mass- und Integrationstheorie in Strukturen. I. Nederl. Akad. Wetensch., Proc. **49**, 167–174 = Indagationes Math. **8**, 64–71 (1946). [MF 16570]

This paper continues the work of the author [Acta Math. **73**, 131–173 (1941); these Rev. **3**, 206], to develop a theory of mass and integration for which the mass distribution is not always of the same sign. The work is based on a structure S of elements $a, b, c, \cdots$, the basic relation among which is "part of," denoted by $\subset$. The operations and relations among them obey all, or all except one, of the axioms of Boolean algebra, the exception being "S contains a greatest element."

A real-valued function $f(x)$ is defined on a subclass of elements of S and is called boundedly additive if (1) for every pair of elements x, y for which f is defined, $f(x)+f(y)=f(x+y)$, and (2) the upper bound of $|f(x)|$ is finite. The positive variation $G(x)$ is the finite upper bound of $f(e)$, $e \subset x$; the negative variation $g(x)$ is the lower bound of $f(e)$. The total variation $T(x)$ is $G(x)+|g(x)|$. In terms of these functions there is defined an outer T-function $T_a(x)$, which is the lower bound of $T(a)$ for all elements a containing x. A set a is T_a-measurable if, for each suitable element w, $T_a(w)=T_a(wa)+T_a(w-wa)$. These functions and measures are then used to define mass functions $T_i(x)$; if $a \subset b$, $T_i(a)=T_a(b)-T_a(b-a)$, $m_f(a)=G_a(a)-|g|_a(a)$.

R. L. Jeffery (Kingston, Ont.).

Ridder, J. Zur Mass- und Integrationstheorie in Strukturen. II. Nederl. Akad. Wetensch., Proc. **49**, 175–184 = Indagationes Math. **8**, 72–81 (1946). [MF 16571]

In this paper the author makes use of product structures based on two structures S^1, S^2 of the type defined in part I [see the preceding review]. Functions similar to those of part I are defined on the elements of the product structures. These functions are then used to define Riemann-Stieltjes and Lebesgue-Stieltjes integrals and completely additive mass functions on the product structures $S^1 \times S^2$.

R. L. Jeffery (Kingston, Ont.).

Milgram, A. N. Bendpoints, geodesics, and free approximations of plane curves. Rep. Math. Colloquium (2) **7**, 37–45 (1946). [MF 16133]

This paper is well summarized by the author in the introduction, essentially as follows. A plane curve is said to bend at the point p if there are arbitrarily small straight line segments bridging p which have only their endpoints on the curve. It is proved that every simple closed curve in the plane has at least three points at which the curve bends only toward its interior. The methods used in the proof are elementary. From this theorem a ready consequence is the fact that between two points in a closed plane region there is at most one geodesic, that is, a shortest arc joining them. If the two points are interior to the region, the geodesic is shown to be the sum of a finite number of convex arcs. Finally, it is proved that, if J is a simple closed curve, then arbitrarily near J (in the sense of Fréchet distance) but lying interior to J there are simple closed polygons of lengths less than the length of J. In addition, it is shown that exterior to J there are polygons arbitrarily near to J whose lengths differ from that of J by an arbitrarily small number, but that only for convex curves do all exterior polygons have a length greater than that of J.

L. H. Loomis (Cambridge, Mass.).

Tsuji, Masatsugu. On Tonelli's theorems on a sequence of rectifiable curves. Jap. J: Math. **17**, 401–410 (1941). [MF 14964]

With the aid of the Parseval relation for Fourier series the author shows that, if $\{C_n\}$ is a sequence of continuous rectifiable curves which tends in the sense of Fréchet to a continuous rectifiable curve C, and if the sequence of corresponding lengths $\{L_n\}$ converges to the length L of C, then with the parametrization $C_n: z=x+iy=F_n(t)$, $C: z=F(t)$, $0 \leqq t \leqq 2\pi$, where for each curve the parameter t is proportional to the arc length along this curve from $z(0)$ to $z(t)$, one has $\int_0^{2\pi} |F'(t)-F_n'(t)|\,dt \to 0$ as $n \to \infty$. This theorem, which is equivalent to one due initially to Tonelli [Fondamenti di Calcolo delle Variazioni, vol. I, Zanichelli, Bologna, 1921, pp. 95–105], is then used to prove the following results of Tonelli [op. cit., pp. 92–94, 186–187] on sequences of functions: if $f(x)$ and $f_n(x)$, $n=1, 2, \cdots$, are continuous functions of bounded variation on $[a, b]$ and $f_n(x) \to f(x)$ uniformly on $[a, b]$, while the sequence of lengths $\{L_n\}$ of the curves $y=f_n(x)$ tends to the length L of $y=f(x)$, then the sequence of derivatives $\{f_n'(x)\}$ converges in measure to $f'(x)$; finally, if in addition $f(x)$ is absolutely continuous on $[a, b]$, then $\int_a^b |f'(x)-f_n'(x)|\,dx \to 0$ as $n \to \infty$. The proof of the result on curves defined parametrically needs slight modifications in order to be rigorous; furthermore, the author gives no references other than to Tonelli's book and is evidently unaware of subsequent literature. In the case of curves defined parametrically a very simple proof of a result equivalent to the first theorem of this paper has been given by McShane in his thesis ["Contributions to the Calculus of Variations, 1930," Chicago University Press, 1931, pp. 195–243, in particular, p. 234]; the last cited result of the paper is an immediate consequence of a theorem of Adams and Lewy [Duke Math. J. 1, 19–26 (1935)] to the effect that convergence in length of $\{f_n(x)\}$ to an absolutely continuous $f(x)$ implies that the total variation of $f_n(x)-f(x)$ on $[a, b]$ tends to zero as $n \to \infty$.

W. T. Reid (Evanston, Ill.).

Cesari, Lamberto. Sulla quadratura delle superficie in forma parametrica. Boll. Un. Mat. Ital. (2) **4**, 109–117 (1942). [MF 16062]

Cesari, Lamberto. Una proprietà caratteristica delle trasformazioni a variazione limitata. Boll. Un. Mat. Ital. (2) **4**, 224–235 (1942). [MF 16073]

Cesari, Lamberto. Sul concetto di trasformazione assolutamente continua. Boll. Un. Mat. Ital. (2) **5**, 5–10 (1943). [MF 16078]

The author is concerned with a continuous transformation $\Phi: x=x(u, v), y=y(u, v)$ with (u, v) in A, the unit square in the (u, v)-plane, and with the transform B of A in the (x, y)-plane. Let K be a square containing B in its interior and let r be a Jordan region in A bounded by a continuous simple curve whose image in the (x, y)-plane is c. If

$O(x, y; c)$ is Kronecker's topological index, the author defines two functions of r as follows:

$$g(r)=\int\int_K |O(x, y; c)|\,dxdy, \qquad u(r)=\int\int_K o(x, y; c)\,dxdy,$$

where $o(x, y; c)=1$ if $O(x, y; c)\neq 0$ and is zero otherwise. Clearly $0\leqq u(r)\leqq g(r)$. He now divides A into a set r_i $(i=1, \cdots, m)$ of Jordan regions which are such that (1) no two have interior points in common and (2) each point of A is in some r_i or on some boundary r_i^*. The images of the curves r_i^* are called c_i. Then $G(\Phi)=\sup \sum g(r_i)$ and $U(\Phi)=\sup \sum u(r_i)$. He also defines two auxiliary functions $\Psi(x, y; \Phi)=\sup \sum |O(x, y; c_i)|$, $\psi(x, y; c)=\sup \sum o(x, y; c_i)$. He then considers the relation of $G(\Phi)$, $U(\Phi)$ to the function

$$W(\Phi)=\int\int_K \Psi(x, y; \Phi)\,dxdy.$$

He defines a branch point of Φ as a point for which $\psi(x, y; \Phi)<\Psi(x, y; \Phi)$ and shows that the totality of such points is denumerable. He next shows that, if Φ is continuous, $U(\Phi)\leqq G(\Phi)\leqq W(\Phi)$, whereas, if Φ is of limited variation, $U(\Phi)=G(\Phi)=W(\Phi)$. It is shown that a continuous surface $S: x=x(u, v), y=y(u, v), z=z(u, v)$ has finite Lebesgue area $L(S)$ if and only if the transformations Φ_1: $x(u, v)$, $y(u, v)$; Φ_2: $x(u, v)$, $z(u, v)$; Φ_3: $y(u, v)$, $z(u, v)$ are of limited variation. In this event

$$W(\Phi_i)\leqq L(S)\leqq W(\Phi_1)+W(\Phi_2)+W(\Phi_3).$$

If $L(S)$ is finite for a continuous surface S, then

$$L(S)\geqq \int\int_A \{J_1^2(u, v)+J_2^2(u, v)+J_3^2(u, v)\}^{\frac{1}{2}}dudv.$$

The author states that the equality holds if and only if the transformations Φ_1, Φ_2, Φ_3 are absolutely continuous in the following sense: (1) for every $\epsilon\geqq 0$ there is a $\sigma>0$ such that $\sum|\pi_i|<\sigma$ implies $\sum g(\pi_i)<\epsilon$, where π_i, $i=1, \cdots, m$, are simple polygonal regions in A without common interior points and (2) $G(\Phi)=\sum G(\pi_i)$. The generalized Jacobians $J_i(u, v)$ appearing above are defined as follows: $J(u, v)=\lim U(q)/|q|$, where q is a square containing (u, v). In the paper on absolute continuity the author gives examples of transformations of limited variation that satisfy the first but not the second part of the definition and that satisfy the second but not the first part. In the paper on limited variation he shows that the equalities $U(\Phi)=G(\Phi)=W(\Phi)$ are valid for Φ continuous. Then he indicates that any absolutely continuous transformation is of limited variation. The paper closes with a characterization of transformations of limited variation. *H. H. Goldstine* (Princeton, N. J.).

Mambriani, A. Su due notevoli integrali del Tonelli. Ann. Mat. Pura Appl. (4) 23, 51–68 (1944). [MF 16611]

Let $f(x, y)$ be a continuous function of (x, y) on the square Q: $0\leqq x\leqq 1$, $0\leqq y\leqq 1$. For fixed y let $V_x(y)$ be the total variation of $f(x, y)$ on the interval $0\leqq x\leqq 1$. Then Tonelli's integral S_x is the (Lebesgue) integral of $V_x(y)$ from $y=0$ to $y=1$; if this diverges, $S_x=\infty$. Similarly for $V_y(x)$ and S_y. Suppose that Q is triangulated by drawing parallels to the x- and y-axes at intervals not greater than δ and λ, respectively, and also one diagonal of each resulting rectangle. Denote by $T_{\delta,\lambda}$ the polyhedron, inscribed in the surface $z=f(x, y)$, (x, y) in Q, whose projection onto the (x, y)-plane yields this triangulation. Let $T_{\delta,\lambda,x}$, $T_{\delta,\lambda,y}$ be Tonelli's integrals for $T_{\delta,\lambda}$. The author shows that S_x and S_y are the lower limits, as δ, λ tend independently to 0, of $T_{\delta,\lambda,x}$, $T_{\delta,\lambda,y}$, respectively. He defines a (more special) inscribed polyhedron $\tau_{\delta,\lambda}$ by an elaborate construction depending on $V_x(y)$, $V_y(x)$; and he shows that, for $\tau_{\delta,\lambda}$, Tonelli's integrals $\tau_{\delta,\lambda,x}$, $\tau_{\delta,\lambda,y}$ converge to S_x, S_y, respectively, as $\delta, \lambda\to 0$. *H. P. Mulholland* (Beirut).

Theory of Functions of Complex Variables

de Groot, J. A theorem concerning analytic continuation. Nederl. Akad. Wetensch., Proc. 49, 213–222 = Indagationes Math. 8, 110–119 (1946). [MF 16575]

If the sequence z_n has the unique limit point z the corresponding "Newton divided difference" interpolation formula is analogous to a power series and may converge to a function regular near z and taking prescribed values at the z_n sufficiently near z. Conversely, if such a function exists the series must converge near z. This converse depends on an extension to the nth difference of the mean value theorem usually ascribed to Weierstrass [cf. Encyklopädie Math. Wiss., vol. 2.1.1 (IIA2), pp. 67–68]. *A. J. Macintyre.*

Ríos, Sixto. On the sets of continuable and noncontinuable Taylor series. Publ. Inst. Mat. Univ. Nac. Litoral 6, 237–245 (1946). (Spanish) [MF 16927]

The author studies the space of power series where the distance between two elements is defined as the reciprocal of the radius of convergence of the difference. The space is metric but not complete, compact or separable; the subspaces of continuable and of noncontinuable power series are both sets all of whose points are interior, except for the zero point in the first case. Corresponding results are indicated for Dirichlet series. [Cf. the author's La Prolongación Analítica de la Integral de Dirichlet-Stieltjes, Madrid, 1944; these Rev. 7, 294.] *R. P. Boas, Jr.* (Providence, R. I.).

Walsh, J. L. On degree of approximation on a Jordan curve to a function analytic interior to the curve by functions not necessarily analytic interior to the curve. Bull. Amer. Math. Soc. 52, 449–453 (1946). [MF 16799]

Let C be a rectifiable Jordan curve, $f(z)$ analytic in the open interior and continuous in the closed interior of C. The author considers a sequence of analytic functions $f_n(z)$ analytic in the open exterior and continuous in the closed exterior of C, allowing poles of $f_n(z)$ exterior to C of total order at most n, with no limit points of these poles on C. Assume that

$$\limsup_{n\to\infty} \{\max |f(z)-f_n(z)|, z\in C\}^{1/n}\leqq\theta<1.$$

Now split $f_n(z)$ into two parts $g_n(z)+h_n(z)$ such that $g_n(z)$ is rational with all its poles exterior to C and $h_n(z)$ is analytic in the open exterior and continuous in the closed exterior of C, $h_n(\infty)=0$. The question is, what does $h_n(z)$ contribute to the degree of approximation? The answer is given by the following inequalities:

$$\limsup_{n\to\infty} \{\max |f(z)-g_n(z)|, z\in C\}^{1/n}\leqq\theta,$$

$$\limsup_{n\to\infty} \{\max |h_n(z)|, z\in C\}^{1/n}\leqq\theta.$$

G. Szegö (Stanford University, Calif.).

Komatu, Yûsaku. Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten. Proc. Phys.-Math. Soc. Japan (3) 25, 1–42 (1943). [MF 15048]

The author develops for the theory of doubly-connected

"Schlitzgebiete" a differential equation corresponding to that of Löwner in the theory of the conformal mapping of simply-connected "Schlitzgebiete." Extremal mappings are studied. The Löwner theory is shown to be a limiting case of the present theory. The author applies his differential equation to Grötzsch's distortion theorem for the conformal mapping of doubly-connected regions and to Teichmüller's work on the Ahlfors distortion theorem. *M. H. Heins.*

Schiffer, Menahem. Sur l'équation différentielle de M. Löwner. C. R. Acad. Sci. Paris 221, 369–371 (1945). [MF 14676]

Let C be a curve in the w-plane with the parametric representation $w=w(\tau)$, $0\leqq\tau\leqq\infty$, $w(\tau)\neq 0$, $w(\infty)=\infty$, where $w(\tau)$ is continuous in τ, and let $C(t)$ be the portion of this curve corresponding to the interval $t\leqq\tau\leqq\infty$. If

$$f(z,t)=\gamma(t)(z+a_2(t)z^2+\cdots),$$

$\gamma(t)>0$, is the function mapping $|z|<1$ onto the w-plane cut along $C(t)$, then, for suitable choice of the parameter τ, Löwner has shown that $f(z,t)$ satisfies the equation

$$\frac{\partial f(z,t)}{\partial t}=z\frac{\partial f(z,t)}{\partial z}\frac{1+k(t)z}{1-k(t)z},$$

where $|k(t)|=1$ and $k(t)$ is continuous in t. The author obtains a differential equation for $k(t)$ in the case when C satisfies $\Im\{w(t)^{-3}(dw(t)/dt)^2P_n[1/w(t)]\}=0$, where $P_n(x)$ is a polynomial of degree n in x. Curves C of this type correspond to extremal schlicht mappings. *D. C. Spencer.*

Alenitzyn, G. On the locally univalent functions. Rec. Math. [Mat. Sbornik] N.S. 18(60), 115–123 (1946). (Russian. English summary) [MF 16679]

P. Montel [Ann. Sci. École Norm. Sup. (3) 54, 39–54 (1937)] considered the class of functions regular in $|z|<1$ and schlicht in any circle of fixed radius ρ, $0<\rho<1$, interior to $|z|<1$. Later S. Gelfer [same Rec. N.S. 8(50), 239–250 (1940); these Rev. 2, 185] considered the class $\{E_\rho\}$ of functions which are schlicht in any circle of fixed radius ρ, $0<\rho<1$, lying in $|z|<1$ and tangent internally to $|z|=1$; moreover, he supposed that $f'(z)\neq 0$ in $0<|z|<1$. The author generalizes further by replacing the tangent circles by lunes B_ρ^α formed of two circular arcs of equal radii which intersect at an angle $\alpha\pi$, $0<\alpha\leqq 1$, and which have a diameter equal to 2ρ, $0<\rho\leqq 1$. One vertex of the lune is supposed to lie on $|z|=1$. A function $f(z)=z^k+\sum_{n=k+1}^{\infty}a_nz^n$, $k\geqq 1$, regular in $|z|<1$, is said to belong to the class $\{E_\rho^\alpha\}$ if there is a number δ, $0<\delta\leqq 2\rho$, such that every point z of the ring $1-\delta<|z|<1$ is located on the diameter of at least one lune B_ρ^α inside which f is schlicht. If $\alpha=1$, the class $\{E_\rho^\alpha\}$ becomes the class $\{E_\rho\}$. Estimates are given for the modulus of the function and for the moduli of the coefficients.

D. C. Spencer (Stanford University, Calif.).

Golousin, G. On the theory of univalent functions. Rec. Math. [Mat. Sbornik] N.S. 18(60), 167–179 (1946). (Russian. English summary)

Let (Σ) denote the class of functions

$$F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$$

which are meromorphic and schlicht in $|\zeta|>1$ and let (S) denote the class of functions $f(z)=z+a_2z^2+\cdots$ which are regular and schlicht in $|z|<1$. The author continues his studies of the mutual behavior of these functions at ζ and $-\zeta$ or z and $-z$ [see the same Rec. N.S. 12(54), 40–47 (1943); these Rev. 5, 93]. Among other results, he proves the following. (i) If $F(\zeta)$ belongs to (Σ), then

$$2(|\zeta|-1/|\zeta|)\leqq|F(\zeta)-F(-\zeta)|\leqq 2(|\zeta|+1/|\zeta|)$$

for $|\zeta|>1$ and equality may be attained on either side. (ii) If $f(z)$ belongs to (S), then

$$\min(|f(z)|,\ |f(-z)|)\leqq|z|/(1-|z|^2),$$

$$|f(z)|+|f(-z)|\leqq\frac{|z|}{(1-|z|^2)}+\frac{|z|}{(1-|z|)^2},$$

$$|f'(z)|+|f'(-z)|\leqq\frac{1+|z|}{(1-|z|)^2}+\frac{|z|}{(1-|z|)^2};$$

the first inequality is precise, as is seen by considering the function $f(z)=z/(1-z^2)$. (iii) If $f(z)=\sum_{n=1}^{\infty}c_nz^n$ belongs to (S) and if $f_1(z)=\sum_{n=1}^{\infty}c_{2n-1}z^{2n-1}$, $f_2(z)=\sum_{n=1}^{\infty}c_{2n}z^{2n}$, then in $|z|<1$ we have

$$|f_k(z)|\leqq\tfrac{1}{2}\{|z|/(1-|z|)+|z|/(1-|z|)^2\},\qquad k=1,2,$$

and $||f_1(z)|-|f_2(z)||\leqq|z|/(1-|z|^2)$. (iv) If $f(z)$ belongs to (S) and if the coefficients of either f_1 or f_2 are $O(n^\alpha)$, $\alpha\geqq 0$, so are the coefficients of the other.

D. C. Spencer (Stanford University, Calif.).

Montel, Paul. Sur les fonctions analytiques dont les valeurs couvrent un domaine d'aire bornée. Publ. Inst. Mat. Univ. Nac. Litoral 6, 273–286 (1946). [MF 16930]

The author studies the family of functions $f(z)$ analytic and single-valued in a given region D and such that the (Euclidean) area $S(f)$ of the Riemannian image of D with respect to $f(z)$ is bounded above by some given positive number $\pi\rho^2$. He distinguishes three types of theorems: (1) of impossibility, (2) of limitation, (3) of extension. An example of the first type is the theorem that if D is $(0<|z|<1)$ and f has an essential singularity at $z=0$, then f cannot belong to the given family. A theorem of the second type is given with $D=(|z|<1)$ and states the explicit sharp bound for $M(r,f)$, $0<r<1$, for f of the family with values fixed at $z=0$. This bound is obtained with the aid of Schwarz's inequality. Similar sharp bounds are given for higher derivatives. For results of the third type the author considers $D=(|z|<R)$ and determines for members of the family with fixed $f(0)$ and $f'(0)$ the largest value of R. Again use is made of Schwarz's inequality.

M. H. Heins (Providence, R. I.).

Milloux, H. Sur une inégalité de M. R. Nevanlinna. Revista Ci., Lima 47, 507–544 (1945). [MF 15433]

Appraisals are given for the complementary term in the second fundamental inequality of the Nevanlinna theory of meromorphic functions. Assuming that the density indices $N(r,\zeta_k)$ associated with the distinct complex values $\zeta_1,\cdots,\zeta_n$ are employed in the formulation of the inequality, one introduces the geometric mean δ of the minimum spherical distances δ_k of ζ_k from the remaining ζ. In terms of δ and other essential parameters of the problem estimates are given for the complementary term. *M. H. Heins.*

Tôki, Yukinari. On the behaviour of a meromorphic function in the neighbourhood of a transcendental singularity. Proc. Imp. Acad. Tokyo 17, 296–300 (1941). [MF 14709]

The author considers a function $f(z)$ which is single-valued and meromorphic for z in a region D with boundary Γ. With a boundary point z_0 three sets are associated. They are: (1) the cluster set $S_{z_0}^{(D)}$ consisting of all a for which

there exists a sequence $\{z_n\}$ with $z_n \varepsilon D$, $n=1, 2, \cdots$, $\lim_{n\to\infty} z_n = z_0$ and $\lim_{n\to\infty} f(z_n) = a$; (2) the cluster set $S_{z_0}^{(D)} = \prod_{n=1}^{\infty} M_n$ where M_n is the closure of the set $\sum S_{z'}^{(D)}$, $0 < |z'-z_0| < 1/n$, $z' \varepsilon \Gamma$; (3) the range $R_{z_0}^{(D)}$ consisting of all a whose antecedents in D with respect to $f(z)$ possess z_0 as a cluster point. The following theorem is demonstrated with the aid of Ahlfors' theory of covering surfaces [Acta Math. 65, 157–194 (1935)]. Under the assumptions listed above, if in addition (i) z_0 is not an isolated point of Γ and (ii) Ω is an arbitrary region of the w-sphere satisfying $\Omega \subset CS_{z_0}^{(D)}$ and $S_{z_0}^{(D)} \cdot \Omega \neq 0$, then $R_{z_0}^{(D)}$ contains all values of Ω save at most two. Applications of this result are made to the theorem of Beurling-Kunugui and allied questions. *M. H. Heins.*

Tumura, Yosiro. Quelques applications de la théorie de M. Ahlfors. Jap. J. Math. 18, 303–322 (1942). [MF 14970]

The author is concerned with extending Ahlfors' theory of covering surfaces [Acta Math. 65, 157–194 (1935)] to the situation where there are given several covering surfaces of a given ground surface. In the finite theory this is accomplished by introducing the sum of the characteristics of the finite covering surfaces and their total number. In the "open" theory, in addition to the Ahlfors condition of regular exhaustibility a second condition is imposed which requires the existence of a regular exhaustion in which the sum of the characteristics and number of components of the approximating surface is bounded above by a given constant times the mean sheet number of the approximant. Application is made to the theory of functions which are meromorphic and single-valued in a given plane region and in addition map the given region into a given region of the w-sphere and respect certain boundary conditions. Related problems for algebroid functions are considered.

M. H. Heins (Providence, R. I.).

***Obrechkoff, Nikola. Quelques classes de fonctions entières limites de polynomes et de fonctions méromorphes limites de fractions rationnelles.** Actualités Sci. Ind., no. 891. Hermann et Cie., Paris, 1941. 47 pp.

In diesem Bericht werden die wichtigsten Ergebnisse samt Beweisen zusammengestellt, die sich im Anschluss von Problemstellungen und Sätzen von Laguerre betreffend in der ganzen z-Ebene konvergenter Folgen von Polynomen mit reellen Nullstellen in den letzten Jahrzehnten ergaben. Insbesondere wird über Arbeiten von Jensen, Lindwart, Montel, Obrechkoff, Petrovitch, Pólya, Saxer und Schur referiert. Das Heft enthält die folgenden drei Kapitel: (1) Fonctions entières limites de polynomes [S. 7–24], (2) Applications diverses [S. 27–33], (3) Fonctions méromorphes limites de fractions rationelles [S. 36–46]. Der Autor hat die Beweise zum Teil gegenüber den Originalabhandlungen erheblich vereinfacht. *W. Saxer* (Zürich).

Myrberg, P. J. Die Kapazität der singulären Menge der linearen Gruppen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 10, 19 pp. (1941). [MF 15210]

The author shows that apart from certain trivial cases the set of singular points of a properly discontinuous group of linear fractional transformations has a positive capacity. This result is based upon an estimate for the Robin constant of a region lying in the extended plane and bounded by a finite number of mutually exterior circles. This estimate is applied to the case of Schottky groups and the general result is obtained from the results for the Schottky case. Applications are made to plane domains which admit nontrivial one-to-one directly conformal maps onto themselves and to automorphic groups of genus zero.

M. H. Heins (Providence, R. I.).

Tsuji, Masatsugu. Theory of conformal mapping of a multiply connected domain. Jap. J. Math. 18, 759–775 (1943). [MF 14983]

Tsuji, Masatsugu. Theory of conformal mapping of a multiply connected domain. II. Jap. J. Math. 18, 977–984 (1943). [MF 14990]

Tsuji, Masatsugu. Theory of conformal mapping of a multiply connected domain. III. Jap. J. Math. 19, 155–188 (1944). [MF 15001]

The intention of the author was to simplify the theory of the conformal mapping of multiply-connected regions. With this in mind he proposed to establish that the capacity of the singular set of a nontrivial Schottky group is zero. In part II he recognized that his proof in part I was not adequate and proposed a second proof. In part III he reported that the proof of part II was also inadequate but expressed the opinion that the theorem (among others) was plausible. However, Myrberg had already shown in the paper reviewed above that the set of singular points of a properly discontinuous group of linear fractional transformations has, apart from certain obvious trivial cases, a positive capacity. *M. H. Heins* (Providence, R. I.).

Wuytack, F. On the fundamental equation of the theory of functions in a special complex algebra. Wis- en Natuurk. Tijdschr. 12, 152–165 (1946). (Dutch) [MF 16749]

L'auteur s'occupe de la théorie des fonctions $f(z)$ d'une variable hypercomplexe $z = \alpha j + \beta k$ ($j^2 = k^2$, $jk = -kj$; l'existence d'une unité à gauche $e = uj + vk$ étant supposée). Il postule que $\int f(z)dz$ ou $\int dz \cdot f(z)$ soit nulle pour chaque chemin fermé, ou que $(\Delta z)^{-1}\Delta f(z)$ ou $\Delta f(z)(\Delta z)^{-1}$ possède une limite pour $\Delta z \to 0$. Ces functions satisfont à des équations différentielles identiques ou analogues à celles de Cauchy-Riemann. L'auteur étudie les opérateurs différentiels respectifs et quelques autres opérateurs qui s'en déduisent. *H. Freudenthal* (Amsterdam).

Theory of Series

Tenca, Luigi. Progressioni aritmetiche contenute in una data progressione aritmetica. Period. Mat. (4) 23, 112–119 (1943).

Wintner, Aurel. A solution theory of the Möbius inversion. Amer. J. Math. 68, 321–339 (1946). [MF 16430]

Let $\mathbf{E} = (\epsilon_{jk})$ and $\mathbf{M} = (\mu_{jk})$ denote the matrices in which, respectively, $\epsilon_{jk} = 1$, $\mu_{jk} = \mu(k/j)$ if k is a multiple of j and the other ϵ_{jk}, μ_{jk} are zero, where $\mu(n)$ is the function of Möbius. The author is concerned with the reciprocal system $\mathbf{E}x = y$, $\mathbf{M}y = x$, that is, the systems of linear equations

$$\text{(E)} \qquad \sum_{m=1}^{\infty} x_{mn} = y_n, \qquad n = 1, 2, \cdots,$$

$$\text{(M)} \qquad \sum_{m=1}^{\infty} \mu(m) y_{mn} = x_n,$$

which play an important role in analytical number theory. While $\mathbf{EM}$ and $\mathbf{ME}$ are both equal to the unit matrix, $\mathbf{M}$

and **E** are not inverse matrices. In fact, **M** is the only left-hand inverse of **E** and a right-hand inverse (but not the only one) and **E** is the only left-hand inverse of **M** and a right-hand inverse (but not the only one). This complicates the solution theory very much. The author calls a solution of (E) regular if $\sum|x_n|<\infty$ and hyperregular if, in addition, the solution is given by (M). There exist regular solutions which are not hyperregular. For the existence of a regular solution $\lim y_n=0$ is necessary, but not convergence of $\sum y_n$; convergence of $\sum|y_n|$ is not sufficient. If $\nu(n)$ is the number of distinct prime divisors of n, then $\sum 2^{\nu(n)}|y_n|<\infty$ is sufficient for the existence of a regular solution. If (E) has a solution such that $\sum 2^{\nu(n)}|x_n|<\infty$, then this solution is hyperregular. Similar results hold for (M). In particular, there can be at most one y such that $\sum 2^{\nu(n)}|y_n|<\infty$. [To the bibliography could be added H. von Koch, Öfversigt K. Vetensk.-Akad. Förh., Stockholm 57, no. 5, 659–668 (1900).] *E. Hille.*

Darevsky, V. On intrinsically perfect methods of summation. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 97–104 (1946). (Russian. English summary)

Let A and B denote regular methods of summability of the Silverman-Toeplitz type. If A is consistent with each B which is not stronger than A, then A is equivalent to convergence. The proof is based on two theorems. If A evaluates a divergent sequence, then it evaluates an unbounded sequence. To each unbounded sequence s_n and number s corresponds a method A which evaluates s_n to s and which evaluates only sequences of the form as_n+t_n, where a is a constant and t_n is a convergent sequence. *R. P. Agnew.*

Kervor, Juan B. On the sum of divergent series. Publ. Inst. Mat. Univ. Nac. Litoral 6, 195–205 (1946). (Spanish) [MF 16923]

Some divergent series of the form

$$x^m-mx^{m+k}+m(m+k)x^{m+2k}-\cdots$$

are evaluated by use of the method of summability defined as follows. If a power series $a_1x+a_2x^2+\cdots$ formally satisfies a linear differential equation and if $f(x)$ is a solution of the differential equation for which $f(0)=0$, then the power series is summable to $f(x)$. The value given for the classic series $1!-2!+3!-4!+\cdots$ is .40367. This method, with calculations involving continued fractions, was used by Euler [Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae 5 (1754–1755), 205–237 (1760)=Opera Omnia, ser. 1, vol. 14, Leipzig-Berlin, 1924, pp. 585–617], who obtained the value .4036526378 ⋯ for the classic series. *R. P. Agnew* (Ithaca, N. Y.).

Lorentz, G. Absolute Konvergenz. Leningrad State Univ. Annals [Uchenye Zapiski] 83 [Math. Ser. 12], 30–41 (1941). (Russian. German summary) [MF 16486]

Let M denote the space of bounded real sequences x_n with norm $\|x_n\|=\text{l.u.b.}\,|x_n|$. Let $L(x_n)$ denote a Banach functional, defined over M, satisfying the conditions (1) $L(ax_n+by_n)=aL(x_n)+bL(y_n)$ whenever a and b are real, (2) $L(x_n)\geqq 0$ whenever $x_n\geqq 0$, (3) $L(x_{n+1})=L(x_n)$, and (4) $L(x_n)=1$ when $x_n=1$. A sequence x_n belongs to class M_1 (is absolutely convergent in the sense of this paper, and the author writes $\text{Lim}\,x_n=p$) if all $L(x_n)$ have the same value p. In order that $x_n\epsilon M_1$, it is necessary and sufficient that

$$\lim_{k\to\infty}(x_{n+1}+x_{n+2}+\cdots+x_{n+k})/k=p$$

uniformly over $n=1, 2, 3, \cdots$. The summability fields of some Silverman-Toeplitz methods of summability are included in M_1, and others include M_1; but none coincides with M_1. *R. P. Agnew* (Ithaca, N. Y.).

Rosenblatt, Alfred. On some Tauberian theorems. Revista Ci., Lima 47, 583–600 (1945). (Spanish)

Let $f_0(x)=\sum_1^\infty a_n^{(0)}x^n$, $f_k(x)=\sum_1^\infty a_n^{(k)}x^n$, $n(n+1)a_n^k=\sum_{r=1}^n ra_r^{(k-1)}$, $k=1, 2, \cdots$. If $f_0(x)\to A$ as $x\to 1$ and if $a_n^{(k)}=o(1/n)$ as $n\to\infty$, then $f_l(x)\to A$ as $x\to 1$ for $l=1, 2, \cdots, k$, $\sum_{r=1}^\infty a_r^{(k)}=A$ and $\sum_{r=1}^\infty a_r^{(k-1)}=A$. The case $k=1$ is a classical theorem of Tauber. *R. P. Boas, Jr.*

Shtshegloff, M. On Poisson's summation. Rec. Math. [Mat. Sbornik] N.S. 18(60), 41–58 (1946). (Russian. English summary) [MF 16676]

The author investigates relations between the limits of indetermination of the partial sums s_n of a series $\sum a_n$ and the limits of indetermination of the Abel means $\sum a_nr^n$. He shows that (i) if $a_n=o(1/n)$, then $\limsup_{n\to\infty}s_n=\limsup_{r\to 1}\sum a_nr^n$, and similarly for lim inf; (ii) if $\lambda_n\uparrow\infty$, $\lambda_{n+1}=O(\lambda_n)$, $a_n=o\{(\lambda_{n+1}-\lambda_n)/\lambda_n\}$, then $\limsup s_n=\limsup_{\epsilon\to+0}\sum a_ne^{-\lambda_n\epsilon}$, and similarly for lim inf. The results are well known [see, for example, J. E. Littlewood, Proc. London Math. Soc. (2) 9, 434–448 (1911), p. 436, where it is observed that the classical proof of Tauber's theorem gives (i); a similar remark applies to (ii)]. *A. Zygmund* (Philadelphia, Pa.).

Amerio, Luigi. Sulla convergenza delle serie doppie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 684–698 (1941).

A method for evaluation of double series, weaker than convergence (Pringsheim) but stronger than absolute convergence, is given. Let $\sum_{i,j=0}^{\infty,\infty}a_{ij}$ be a double series with partial sums $s_{mn}=\sum_{i,j=0}^{m,n}a_{ij}$. The series is Amerio summable to S if to each $\epsilon>0$ corresponds an integer r such that $|S-\Sigma'|<\epsilon$ whenever Σ' is a sum of terms u_{ij} which is such that (1) Σ' contains u_{rr} and (2) whenever Σ' contains u_{mn}, it contains all terms in s_{mn}. Each series which is Amerio summable has bounded partial sums of the type Σ', and is convergent by rows and by columns. Theorems on Cauchy products and power series for simple series are extended to double series which are Amerio summable.

This paper antedates work of Sheffer [Amer. Math. Monthly 52, 365–376 (1945); these Rev. 7, 13] where the same method of summability was defined and studied. Criticisms and a reply are given in the two papers reviewed below. *R. P. Agnew* (Ithaca, N. Y.).

Cesari, Lamberto. Sulla convergenza delle serie doppie. Ann. Scuola Norm. Super. Pisa (2) 11, 133–150 (1942). [MF 16758]

This paper treats summability and restricted summability of double series by the Cesàro arithmetic mean method $C(1, 1)$ and by the Abel (or Euler or Poisson) power series method P. [The invention and the application to Fourier series of restricted convergence and summability are due to C. N. Moore (1912) [see his "Summable Series and Convergence Factors," Amer. Math. Soc. Colloquium Publ., vol. 22, New York, 1938], not to Tonelli (1927) as the author says.]

Let $\sum a_{mn}$ be a double series with partial sums s_{mn}. If $\sum a_{mn}$ converges to S and if

(1) $$\lim_{m+n\to\infty}a_{mn}=0,$$

then $\sum a_{mn}$ is restrictedly summable P to S. If $\sum a_{mn}$ converges to S and if there exist five positive constants A, B, C, b, c such that $b<1$, $c<1$ and

$$(2) \qquad |s_{mn}| < A + B\left(\frac{m+1}{n+1}\right)^b + C\left(\frac{n+1}{m+1}\right)^c, \quad m, n \geqq 0,$$

then $\sum a_{mn}$ is restrictedly summable P to S. If $\sum a_{mn}$ is summable $C(1, 1)$ to S and (1) holds, then $\sum a_{mn}$ is restrictedly summable P to S. If $\sum a_{mn}$ is restrictedly summable $C(1, 1)$ to S, and if there exist five positive constants A, B, C, b, c such that $b<2$, $c<2$, and

$$(3) \qquad \left|\frac{1}{(m+1)(n+1)} \sum_{j,k=0}^{m,n} s_{jk}\right| < A + B\left(\frac{m+1}{n+1}\right)^b + C\left(\frac{n+1}{m+1}\right)^c,$$

$m, n \geqq 0$, then $\sum a_{mn}$ is restrictedly summable P to S.

A direct proof is given of a corollary of the following theorem which is an immediate consequence of G. M. Robison's treatment [Trans. Amer. Math. Soc. 28, 50–73 (1926)] of summability of double series. If $\sum a_{mn}$ is convergent to S and has bounded partial sums, then $\sum a_{mn}$ is summable P to S. The corollary is stronger than a theorem of Amerio [see the preceding review] and the author says that this shows the uselessness of the nonregular method of summability introduced by Amerio and, independently, by Sheffer [Amer. Math. Monthly 52, 365–376 (1945); these Rev. 7, 13]. *R. P. Agnew* (Ithaca, N. Y.).

Amerio, Luigi. Ancora sulla convergenza delle serie doppie. Boll. Un. Mat. Ital. (2) 5, 174–181 (1943). [MF 16104]

Two thirds of the present paper review and compare the papers of Amerio and Cesari reviewed above, the author insisting that there is virtue in his new method. The remainder (which does not use the new method) proves the following theorem on Cauchy products of double series. If $\sum a_{mn}$ converges by rows to A, and if $\sum b_{mn}$ converges to B and converges absolutely, then the Cauchy product-series $\sum c_{mn}$ converges by rows to AB. *R. P. Agnew.*

Fourier Series and Generalizations, Integral Transforms

Aprile, Giuseppe. Il teorema di Fourier. Period. Mat. (4) 22, 66–69 (1942).

Lu, Yun Kai. A note on a theorem of Turán. Duke Math. J. 13, 129–131 (1946).

Let $f(x)$ be an odd function of period 2π which is non-negative and convex in $(0, \pi)$. Let $S_n^{(\alpha)}$ be the nth Cesàro mean of order α of the Fourier series of $f(x)$. It is known [Turán, J. London Math. Soc. 10, 277–280 (1935)] that $f(x) \geqq S_n^{(\beta)} \geqq S_n^{(\alpha)} > 0$ if α and β are positive integers and $\alpha \geqq \beta$. The present paper shows that the inner inequality no longer holds if α and β are not integers. *P. Civin.*

Kuttner, B. Note on the Gibbs phenomenon. J. London Math. Soc. 20, 136–139 (1945).

The author considers a generalized form of the Gibbs phenomenon for methods of summability termed K-methods by Hardy and Rogosinski [Fourier Series, Cambridge University Press, 1944; these Rev. 5, 261]. For a given K-method let $t_n(\theta)$ denote the means of the Fourier series of a function $f(\theta)$. The means are said to exhibit a generalized Gibbs phenomenon at the point $\theta = x$ if the interval

$$[\liminf_{\delta \to x,\, n \to \infty} t_n(\theta), \quad \limsup_{\delta \to x,\, n \to \infty} t_n(\theta)]$$

contains points outside the interval

$$[\liminf_{\delta \to x} f(\theta), \quad \limsup_{\delta \to x} f(\theta)].$$

The extension comes from the inclusion of functions $f(\theta)$ other than those having a simple jump at $\theta = x$.

A necessary and sufficient condition that a K-method should have means which do not display the generalized Gibbs phenomenon is that its kernel $K_n(t)$ is bounded below. For (C, r) summability this asserts that, if $r<1$, a function exists which displays the phenomenon, while for $r \geqq 1$, no such function exists. The comparable result for the usual Gibbs phenomenon is that of Cramér [Ark. Mat. Astr. Fys. 13, no. 20 (1919)], which asserts that at an ordinary jump of any function $f(\theta)$ the (C, r) means display the Gibbs phenomenon if $r < .439 \cdots$, but not otherwise.

P. Civin (Eugene, Ore.).

Natanson, I. Quelques applications de l'intégrale de Vallée-Poussin à la théorie des séries de Fourier. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 393–395 (1945).

Let $V_n(x)$ be the singular integral of de la Vallée Poussin, $V_n(x) = ((2n)!!/(2n-1)!!2\pi)\int_{-\pi}^{\pi} f(t) \cos^{2n}\frac{1}{2}(t-x)\,dt$. In a previous paper [same C. R. (N.S.) 45, 274–277 (1944); these Rev. 6, 267] the author considered the convergence of $V_n(x)$ to $f(x)$ when $f(x)$ is continuous. He now shows that the method of summability with means $V_n(x)$ is stronger than $(C, 2)$ summability. The usual necessary and sufficient conditions that a function be continuous, bounded L^p or L [Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, pp. 79–87] are stated in terms of the means $V_n(x)$.

P. Civin (Eugene, Ore.).

Chandrasekharan, K. On the summation of multiple Fourier series. III. Bull. Amer. Math. Soc. 52, 474–477 (1946). [MF 16804]

Denote by $s_n(x)$ the spherical sums of the k-tuple Fourier series $f(x) \sim \sum a_{\nu_1 \cdots \nu_k} \exp\{i(\nu_1 x_1 + \cdots + \nu_k x_k)\}$. The main result of the paper is: assume

$$(*) \qquad \sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1 \cdots \nu_k}|^2 < \infty;$$

then Abel summability of the sequence s_n implies its convergence. Combined with a result of S. Bochner [Trans. Amer. Math. Soc. 40, 175–207 (1936)] it follows that (*) implies convergence of the sequence $s_n(x)$ at each point of continuity. The reviewer remarks that (*) can be replaced by the more general assumption

$$\lim_{h \to 0} \limsup_{n \to \infty} \sum |a_{\nu_1 \cdots \nu_k}| = 0,$$

where $\sum$ extends over $x \leqq \nu_1^2 + \cdots + \nu_k^2 \leqq (1+\delta)x$.

O. Szász (Cincinnati, Ohio).

Sunouchi, Gen-ichirô. On the strong summability of series of orthogonal functions. Proc. Imp. Acad. Tokyo 20, 251–256 (1944). [MF 14889]

By the use of standard methods slight simplifications and minor extensions of well-known results are obtained.

M. Kac (Ithaca, N. Y.).

Nachbin, Leopoldo. On linear expansions. I. Trans. Amer. Math. Soc. 59, 437–440 (1946). [MF 16466]

Nachbin, Leopoldo. On linear expansions. II. Summa Brasil. Math. 1, 17–20 (1946). (English. Portuguese summary) [MF 15866]

Let the family $\{\phi_n(x)\}$ of measurable functions be defined in a set E of positive measure. Generalizing a theorem of

Stone [Ann. of Math. (2) 32, 233–238 (1931)] the author proves the equivalence of the two following properties: (I) the convergence of $\sum c_n\phi_n(x)$ almost everywhere in E implies $c_n=o(1)$; (II) the convergence of $\sum|c_n\phi_n(x)|$ almost everywhere in E implies $\sum|c_n|<\infty$. *R. Salem.*

Boas, R. P., Jr., and Pollard, Harry. Properties equivalent to the completeness of $\{e^{-t}t^{\lambda_n}\}$. Bull. Amer. Math. Soc. 52, 348–351 (1946). [MF 16204]

Let $\{\lambda_n\}$ be a monotone increasing sequence of positive integers. The following three propositions are equivalent: (a) a_n complex and $O(n^\beta)$ for some β and $\Delta^{\lambda_n}a_0=0$ implies that a_n is a polynomial in n; (b) $\{t^{\lambda_n}e^{-t}\}$ is complete in $L_2(0,\infty)$; (c) $f(z)$ regular and $O(|z|^\alpha)$, for some α, in $x>-\epsilon$ and $f^{(\lambda_n)}(0)=0$ ensure that $f(z)$ is a polynomial. The implications (a)↔(b) (and a natural necessary and sufficient condition for (a)) are due to Fuchs, but the authors' proofs are less involved. The most original part of the note is in the relation (c)→(a). The proof consists in showing that $|f(z)|=O(|z|^\beta)$ in a left half plane overlapping $x>-\epsilon$ by making use of the expansion in powers of $z/(1-z)$ to continue $f(z)$. It would be of interest to replace the chain of results leading from (b) for $\{\lambda_n\}$ to (b) for $\{\mu_n\}$, $\mu_n=\lambda_n-N$, by a direct verification. *D. G. Bourgin.*

González Domínguez, A. On certain inversion formulas. Publ. Inst. Mat. Univ. Nac. Litoral 6, 207–214 (1946). (Spanish) [MF 16924]

The author considers the series $\sum_{n=1}^{\infty}c_nz/(z^2+n^2)$ and gives five expressions for the coefficients c_n, which are assumed to be real. One type of expression is illustrated by the following theorem. If $\sum_{n=1}^{\infty}|nc_n|<\infty$, then

$$2\pi^2ic_n=\int_0^{2\pi}\cos nt\,dt\int_{x-i\infty}^{x+i\infty}e^{zt}f(z)dz,$$

where x is an arbitrary positive number. Other types of expressions employ series of Laguerre polynomials and higher order derivatives with respect to t, instead of integrals with respect to t. *I. Opatowski* (Ann Arbor, Mich.).

Fraser, W. C. G. Inversion formulae for the factorial transform. Duke Math. J. 13, 239–246 (1946).

It is shown that the transform $f(s)=\int_0^\infty\beta(s,t+1)d\alpha(t)$ is inverted by the formula

$$\alpha(t)=\lim_{r\to\infty}\sum_{n=1}^{\infty}\left\{\lim_{s\to\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+s}}{m!n!}ns\epsilon^{mrt}(1-\epsilon^{-mr})^{-ns}f(ns)\right\}$$

at all points of continuity of $\alpha(t)$. This is valid provided only that $\sigma_c<\infty$. A second real inversion formula is obtained under the hypothesis $\sigma_a<1$. *H. Pollard.*

Ferrari, María Angélica. Properties of the D_λ transformation. Publ. Inst. Mat. Univ. Nac. Litoral 6, 321–327 (1946). (Spanish) [MF 16934]

Repetto, Celina. Uniform convergence and inversion of D integrals in the elliptic and parabolic complex field. Publ. Inst. Mat. Univ. Nac. Litoral 6, 315–320 (1946). (Spanish) [MF 16933]

Both papers contain only statements of results. The transform is (*) $f(z)=\int_0^\infty e^{-z\lambda(t)}dA(t)$, where $\lambda(t)$ is nondecreasing. When z is an ordinary complex variable, the properties of (*) are exact analogues of those of the ordinary Laplace transform. The first paper considers mainly properties of $f(z)$ in the complex plane and special sufficient conditions for representability of a function in the form (*). The second paper considers particularly sufficient conditions for uniform convergence. Both papers give complex inversion formulas. The second paper also considers the case where z is a "parabolic" complex variable ($z=x+jy$, $j^2=0$). The novel point of the theory is that the abscissas of uniform and absolute convergence are equal. [Some of the results of the first paper are stated with incomplete hypotheses; for example, property (9) is false as stated.]

R. P. Boas, Jr. (Providence, R. I.).

Poli, L. Sur deux règles du calcul symbolique. Ann. Univ. Lyon. Sect. A. (3) 7, 21–29 (1944).

Some new formulas in the symbolic calculus are obtained by integrating or differentiating with respect to a parameter inside a Laplace integral. *H. Pollard* (Ithaca, N. Y.).

Følner, Erling. On the structure of generalized almost periodic functions. Danske Vid. Selsk. Math.-Fys. Medd. 21, no. 11, 30 pp. (1945). [MF 15400]

The author gives a structural characterization of a certain class of a.p. (almost periodic) functions previously defined by A. S. Kovanko [C. R. Acad. Sci. Paris 198, 792–794 (1934); Rec. Math. [Mat. Sbornik] 42, 3–10 (1935)] in terms of approximating polynomials. He also studies the relationship of K-a.p. functions to Besicovitch's B^p-a.p. functions, and is thus led (for each $p\geqq1$) to a class of functions equivalent to the class B^p in the sense of having the same Fourier series, but having a simpler structural characterization than the class B^p. *R. H. Cameron.*

Polynomials, Polynomial Approximations

Cypkin, Ya. Z., and Bromberg, P. V. On the degree of stability of linear systems. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1945, 1163–1163 (1945). (Russian)

The authors transform the well-known Hurwitz criterion for the roots of a polynomial to be in the half-plane $x<0$ in order to obtain a criterion for the roots to be in $x<-\delta<0$. The case of a cubic is discussed in detail and the results are presented graphically. *R. P. Boas, Jr.*

Couffignal, Louis. Recherches de mathématiques utilisables. Sur les conditions de stabilité des systèmes oscillants. Revue Sci. 83, 195–210 (1945).

The author gives a simplified form of Routh's criterion for an algebraic equation with real coefficients to have negative real parts for all of its roots. The details are worked out for an equation of the eighth degree. A chart is given which enables one with very little calculation to apply the test to any equation of degree 8, 6, 4 or 2.

P. Franklin (Cambridge, Mass.).

Meerov, M. V. Criterion for aperiodicity of a regulator. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1945, 1169–1178 (1945). (Russian)

The author gives a determinantal criterion for a polynomial to have real roots and illustrates it for a quartic.

R. P. Boas, Jr. (Providence, R. I.).

Rémès, E. J. On the construction of mean-power approximations in the complex domain. Praci Sičnevoï Sesiï Akad. Nauk URSR. Dopovidi Viddilu Fiz.-Him. Mat. Nauk **2**, 207–214 (1944). (Ukrainian. Russian and English summaries)

In an earlier paper [Rec. Math. [Mat. Sbornik] N.S. 9(51), 437–450 (1941); these Rev. **2**, 363] the author described a method for constructing effectively the polynomial of best approximation, in the sense of $2k$th powers, to a given real-valued continuous function. He now extends the method to complex-valued functions. The correction term in the author's approximation algorithm converges more slowly to zero than in the real case. *R. P. Boas, Jr.*

Merli, L. Recenti risultati sulla convergenza dei polinomi di interpolazione di Lagrange e di Hermite. Giorn. Ist. Ital. Attuari **11**, 107–118 (1940). [MF 16626]

Expository article describing results due to Bernstein, Faber, Fejér, and others. *A. Erdélyi* (Edinburgh).

Bottema, O. A relation for the polynomials of Laguerre and Hermite. Nederl. Akad. Wetensch., Proc. **49**, 65–71 = Indagationes Math. **8**, 29–35 (1946). (Dutch) [MF 16564]

The main result,

$$\sum_{n=0}^{\infty}\binom{n+\alpha+\beta}{n}L_n^{(\alpha)}(x)t^n=(1-t)^{-\alpha-1}e^{-xt/(1-t)}L_m^{(\beta)}(xt/(1-t)),$$

is not new [cf., for example, Erdélyi, Math. Z. **42**, 641–670 (1937), equation (5.7)]. For $\alpha=\pm\frac{1}{2}$ there follow corresponding formulae for even and odd Hermite polynomials. From the series a number of (known) integral relations are deduced by means of the orthogonal properties of Laguerre and Hermite polynomials. *A. Erdélyi* (Edinburgh).

Differential Equations

***Cartan, Élie. Les systèmes différentiels extérieurs et leurs applications géométriques.** Actualités Sci. Ind., no. 994. Hermann et Cie., Paris, 1945. 214 pp.

The first part of this book contains the theory of integration of total differential equations connected with a general system of exterior differential forms (covariant alternating quantities). The symbolism used is the ω-method introduced in Cartan's well-known publications [Ann. Sci. École Norm. Sup. (3) **18**, 24–311 (1901); **21**, 153–206 (1904), in particular, chap. I] with some modifications due to E. Kähler [Einführung in die Theorie der Systeme von Differentialgleichungen, Hamburger Math. Einzelschr., no. 16, Teubner, Leipzig-Berlin, 1934]. The first two chapters contain an exposition of the method. In chapter III, after introducing the important notions of closed systems and characteristic systems the theory of completely integrable systems is presented and applied to the ordinary problem of Pfaff. Chapter IV contains the definitions of the integral elements, the characters and the genus and two fundamental existence theorems. Systems in involution are defined in chapter V and this chapter contains several simple forms of the conditions for these systems. The theory of prolongation is dealt with in chapter VI. For the chief theorem of prolongation, proved by Cartan in 1904 [loc. cit.], another proof is given for the case $m=2$. In § 117 special attention is paid to the cases where the proof is not valid.

The second part of the book contains applications to several problems of differential geometry. Chapter VII deals with old and new problems of the classical theory of surfaces. In each case the degree of freedom of the solution is discussed. The last chapter contains problems with more than two independent variables. A new method is developed for orthogonal systems in n variables. The problem of the realization of a V_2 with a given ds^2 in an R_4 is discussed elaborately and special attention is paid to the singular solutions of this problem. *J. A. Schouten* (Epe).

Gillis, Paul. Sur les formes différentielles et la formule de Stokes. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **20**, no. 3, 95 pp. (1943). [MF 16506]

This paper gives proofs of results stated elsewhere by the author [Bull. Soc. Roy. Sci. Liége **11**, 464–470 (1942); these Rev. **7**, 120]. *F. G. Dressel* (Durham, N. C.).

Sato, Tunezo. On the boundary problem of

$$\frac{d}{dx}\left(x^m\frac{dy}{dx}\right)+\lambda x^n y=0.$$

Proc. Phys.-Math. Soc. Japan (3) **24**, 640–656 (1942). [MF 15036]

The paper deals with the problem of the title, where $m\geqq 0$ and $y'(0)=y(1)=0$. The equation is explicitly solvable by Bessel functions. The existence of a solution may thus be investigated by explicit calculation. It is found that there exists a solution (unique, except for a constant factor) for characteristic values of λ when $n+1-m>0$. The expression of the solution in Bessel functions is different in the cases $1>m\geqq 0$ and $m\geqq 1$. The Green's function of the boundary problem is calculated. The equivalent Fredholm integral equation is given and its Fredholm determinant is written out. An application to the torsional vibration of a beam is indicated. *R. E. Langer* (Madison, Wis.).

Sato, Tunezo. On Green's function of linear differential equation of the fourth order. Proc. Phys.-Math. Soc. Japan (3) **25**, 667–672 (1943). [MF 15078]

It is observed that the flexural vibration of a bar is subject to formulation as a differential system

$$y^{(iv)}+ay''+\lambda y=0,\quad y(0)=y(1)=y''(0)=y''(1)=0.$$

By simple considerations of Green's functions this boundary problem is expressed as an integral equation. From this the characteristic values of λ are read off. *R. E. Langer.*

Štokalo, Ĭ. Z. Criteria for stability and instability of the solutions of linear differential equations with quasi-periodic coefficients. Akad. Nauk URSR. Informaciĭnĭ Byuleten no. 1(10–11), 38–39 (1945). (Ukrainian and Russian) [MF 16227]

Štokalo, Ĭ. Z. Linear differential equations of the nth order with quasiperiodic coefficients. Akad. Nauk URSR. Informaciĭnĭ Byuleten no. 1(10–11), 40–42 (1945). (Ukrainian and Russian) [MF 16228]

Štokalo, Ĭ. Z. Systems of linear differential equations with quasiperiodic coefficients. Akad. Nauk URSR. Informaciĭnĭ Byuleten no. 1(10–11), 42–45 (1945). (Ukrainian and Russian) [MF 16229]

Štokalo, Ĭ. Z. Generalized Gibbs formula for the case of linear differential equations with variable coefficients. Akad. Nauk URSR. Informaciĭnĭ Byuleten no. 1(10–11), 46–50 (1945). (Ukrainian and Russian) [MF 16230]

These are summaries of papers which are set up in type

but apparently not yet published. In these papers criteria for stability of solutions of systems of differential equations as well as explicit formulas for solutions of systems with right members are extended from the familiar case of linear systems with constant coefficients to the much more complicated case where the coefficients involve trigonometric functions of the independent variable. The analogues of formulas of operational calculus as well as the Bromwich integral reappear in the generalized case. *N. Levinson.*

Chiellini, Armando. Alcune ricerche sulla forma dell'integrale generale dell'equazione differenziale del primo ordine $y'=C_0y^3+3C_1y^2+3C_2y+C_3$. Rend. Sem. Fac. Sci. Univ. Cagliari **10**, 16–28 (1940). [MF 16214]

A necessary and sufficient condition that the general solution of (1) $y'=C_0y^3+3C_1y^2+3C_2y$ (where the C's are functions of x) can be put in the form

(2) $$(y_1-y)^{\alpha_1}(y_2-y)^{\alpha_2}(y_3-y)^{\alpha_3}=Cy_1^{\alpha_1}y_2^{\alpha_2}y_3^{\alpha_3},$$

C constant, with $\alpha_1+\alpha_2+\alpha_3=0$, α's constant, is the existence of three particular solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, such that $\alpha_1y_1+\alpha_2y_2+\alpha_3y_3\equiv 0$; or, if y_1 and y_2 are two particular solutions, then for proper choice of constants α_1 and α_2, $y_3=(\alpha_1y_1+\alpha_2y_2)/(\alpha_1+\alpha_2)$ is a solution.

A necessary and sufficient condition that the general solution of (1) can be put in the form (2) is the existence of three particular solutions y_1, y_2, y_3 such that

(3) $$C_0(y_1+y_2+y_3)+3C_1=0.$$

For the equation $y'=C_0y^3+3C_1y^2+3C_2y+C$ it is shown that knowledge of four particular solutions, three of which satisfy (3), enables one to put the general solution in the form

$$\frac{(y_1-y)^{\alpha_1}(y_2-y)^{\alpha_2}(y_3-y)^{\alpha_3}}{(y_1-y_4)^{\alpha_1}(y_2-y_4)^{\alpha_2}(y_3-y_4)^{\alpha_3}}=C.$$

If (3) is solvable by quadrature, its general solution can be put in the form

$$(z-z_1)^{\alpha_1}(z-z_2)^{\alpha_2}(z-z_3)^{\alpha_3}=HK\exp\{\tfrac{3}{2}a\int(p_0p_2^2)^{\frac{1}{3}}dx\},$$

K constant, where z refers to the transformed equation $z'=p_0z^3+3p_2z+p_3$ obtained from (3) by use of the substitution $y=z-C_1/C_0$ and a is a constant occurring in a condition for solvability by quadrature, established in a paper by the author, referred to as being in press in Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. *A. B. Brown.*

Chiellini, A. Studio del sistema differenziale

$$p^2+q^2=A(xy),\quad r+t=0.$$

Rend. Sem. Fac. Sci. Univ. Cagliari **12**, 53–68 (1942). [MF 16220]

(I) To determine, under an inversion in a circle in the plane, what curves, other than rays and concentric circles, are transformed into curves such that, for the curve and its transform, the curvatures at corresponding points are proportional. The only such curves are rays through the center and concentric circles. If the inversion is replaced by a general conformal transformation, the problem is shown, in general, to lead to a differential equation of the second or third degree, according as the constant of proportionality is or is not zero.

(II) In studying the problem of part (I), the following system of differential equations arises:

(1) $$(\partial u/\partial x)^2+(\partial u/\partial y)^2=A(x,y),\quad \partial^2u/\partial x^2+\partial^2u/\partial y^2=0,$$

where $A(x,y)$ is a given function. Let $B=A^{-2}(x,y)$. Then the condition $B_x^2+B_y^2=B(B_{xx}+B_{yy})$ is shown to be necessary and sufficient that (1) be completely integrable with the maximum number of arbitrary constants, that is, two. Other results and applications are given. *A. B. Brown.*

Grinberg, G. A. A new method of solving certain boundary problems for equations of mathematical physics permitting a separation of variables. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **10**, 141–168 (1946). (Russian)

The author considers a second-order linear differential equation $D(s_1,\cdots,s_m)n=f(s_1,\cdots,s_m)$ and seeks to find a solution u satisfying given boundary conditions for $s_k=a_k, b_k$ $(k=1,\cdots,m)$. He assumes that D is such that the homogeneous equation $Du=0$ has a solution $u=Q(s_1)F(s_2,\cdots,s_m)$, where $Q(s_1)$ satisfies a suitable second-order (ordinary) differential equation. Then the coefficients in the development of a solution of the original boundary-value problem as a function of s_1, in terms of characteristic functions of the equation for $Q(s_1)$, satisfy a differential equation in the remaining $m-1$ variables. If $m>2$ and the new equation is of the proper character, the process continues. The author discusses conditions for the validity of this formal process and illustrates it with examples. *R. P. Boas, Jr.*

Tsuji, Masatsugu. On the boundary value of a harmonic function in space. Jap. J. Math. **19**, 111–137 (1944). [MF 14999]

Let Δ be a domain in 3-space bounded by a closed surface S consisting of a finite number of smooth pieces separated by a finite set of piecewise smooth Jordan arcs L. It is also assumed that S satisfies the two conditions (i) if S^δ is the portion of S within a distance $\delta>0$ of L, there exists $a(\delta)>0$ such that each point of $S-S^\delta$ admits a tangent sphere lying inside Δ of radius at least $a(\delta)$, (ii) a_0 exists such that each point of $S-L$ admits a tangent sphere lying outside Δ of radius at least a_0. The author considers functions harmonic in Δ and develops the theory of their boundary values with the aid of a Green's function and the Stieltjes integral; the methods and theorems are analogous to those found in chapters II and III of G. C. Evans, The Logarithmic Potential [Amer. Math. Soc. Colloquium Publ., v. 6, New York, 1927], which treats functions harmonic in a circle. The results obtained include Fatou's theorem for Δ: if u is a positive harmonic function in Δ, $\lim u(Q)$ exists for nontangential approach of Q to P for almost all P in S; they also include necessary and sufficient conditions for the representation of harmonic functions in Δ by the Green's function and Stieltjes or Lebesgue integrals in terms of the uniform bounded variation or uniform absolute continuity of $\int u dS$, the integral being extended over a set on a surface parallel to S. *J. W. Green* (Los Angeles, Calif.).

Brelot, Marcel. Minorantes sous-harmoniques, extrémales et capacités. J. Math. Pures Appl. (9) **24**, 1–32 (1945). [MF 15868]

The author coordinates and generalizes some of his earlier theories and related work by others. Most of the discussion is formulated for the compact space $\hat{R}_\tau$ of $\tau\geqq 2$ dimensions previously studied by the author [Ann. Sci. École Norm. Sup. (3) **61**, 301–332 (1944); these Rev. **7**, 204]. In an open set Ω of $\hat{R}_\tau$ let the function f have an upper bound locally and a subharmonic minorant. Then the family Σ of functions which are subharmonic in Ω and nowhere exceed f have an upper envelope Σ_f which is quasi-subharmonic,

that is, subharmonic except possibly on a polar set. Also, the family S of functions subharmonic in Ω and exceeding f at most on a polar set have an upper envelope S_f called the extremal, which is the largest function in S. These propositions are generalizations of results due to N. Sjöberg [Neuvième Congrès Math. Scandinaves, 1938, Helsingfors, 1939, pp. 309–319]. Fundamental properties of Σ_f and S_f are discussed.

Given in $\bar{R}_r$ an open set Ω (which has a nonpolar complement) and a function $u \leqq 0$ subharmonic in Ω, the extremal of u for a set E in Ω is the function S_f formed for $f=0$ on E and $f=u$ on $\Omega-E$. This extremalization is a generalization of the sweeping out process. It has been studied in a less general form by the author [C. R. Acad. Sci. Paris **206**, 35–37 (1938); Acad. Roy. Belgique. Bull. Cl. Sci. **25**, 125–137 (1939); J. Math. Pures Appl. (9) **19**, 319–337 (1940); these Rev. **1**, 238; **3**, 47] and by A. F. Monna [Nederl. Akad. Wetensch., Proc. **42**, 745–752 (1939); **43**, 81–86, 87–90, 497–511 (1940); **44**, 50–61 (1941); these Rev. **1**, 122, 238; **2**, 58, 93]. A considerable part of the paper is devoted to the properties of this extremal and related concepts, including interior and exterior extremals. The author also includes a discussion of generalized capacities and a section on applications. *F. W. Perkins.*

Brelot, Marcel. Sur la mesure harmonique et le problème de Dirichlet. Bull. Sci. Math. (2) **69**, 153–156 (1945).

This paper is devoted to several examples illustrating instances of remarkable behavior of harmonic functions in the neighborhood of boundary points of their region of definition. The functions studied include the harmonic measure of suitably chosen point sets. *F. W. Perkins.*

Kametani, Syunzi. On some properties of Hausdorff's measure and the concept of capacity in generalized potentials. Proc. Imp. Acad. Tokyo **18**, 617–625 (1942). [MF 14788]

Kametani, Syunzi. On Hausdorff's measures and generalized capacities with some of their applications to the theory of functions. Jap. J. Math. **19**, 217–257 (1945). [MF 15004]

Let Ω be any separable metric space with distance $\rho(p, q)$ and let $m_h(E)$ be Hausdorff measure in Ω based on the function $h(r)$. Let μ be a distribution of positive mass on a Borel subset E of Ω and let $\phi(r)$ be a strictly decreasing continuous function such that $\lim_{r\to 0} \phi(r) = +\infty$. The author considers the generalized potential $u(p) = \int_E \phi(\rho(p, q))d\mu(q)$ and the generalized capacity $C^\phi(E)$, or ϕ-capacity, defined by

$$C^\phi(E) = \phi^{-1}\left[\inf_\mu \left\{\sup_{p\in\Omega} u(p)\right\}\right],$$

where μ is of total mass 1 on E, and $\phi^{-1}(+\infty) = 0$. For a set A, not necessarily a Borel set, $C^\phi(A) = \sup C^\phi(E)$ for $E \subset A$. If Ω is Euclidean 2-space, the cases $h(r) = \pi r^2/4$, $h(r) = r^\alpha$ $(\alpha > 0)$, $h(r) = (\log 1/r)^{-1}$ yield Lebesgue plane measure, α-dimensional measure (Carathéodory linear measure if $\alpha = 1$) and logarithmic measure, respectively; $\phi(r) = \log 1/r$, $\phi(r) = 1/r$ yield the logarithmic and Newtonian potentials, respectively, with the corresponding capacities.

The principal result of the first paper is the following theorem: if a set A is of positive ϕ-capacity, it is of infinite h-measure, where $h(r) = 1/\phi(r)$. This theorem was conjectured by Nevanlinna in the case of plane logarithmic measure and subsequently proved by Erdös and Gillis [J. London Math. Soc. **12**, 185–192 (1937)]. In the case of Carathéodory linear measure it was proved by T. Ugaheri [Proc. Imp. Acad. Tokyo **18**, 602–605 (1942); these Rev. **7**, 279]. The proof depends on a general density theorem obtained by the author: if E is any set of finite h-measure, then the subset of points of E, on which the upper density of E is less than 1, is of h-measure zero.

Part I of the second paper consists essentially of the material in the first, somewhat enlarged and elaborated, and in addition a section on n-dimensional Euclidean space. This section contains the following special result: a bounded Borel set of finite 1-dimensional measure is of Newtonian capacity zero and a bounded Borel set of positive $(1+\delta)$-dimensional measure is of positive Newtonian capacity if $\delta > 0$.

Part II of the second paper consists of applications to the theory of functions, in particular, to the study of what are known as W-sets. A compact set E is a W-set provided there exists no function which is regular, bounded, and nonconstant in the connected portion of CE containing ∞. A W-set is necessarily totally disconnected. The author studies the characterization of W-sets in terms of Hausdorff measures and related capacities. A number of results are obtained, the chief of which may be summarized as follows. A compact set of 1-dimensional measure zero is a W-set. A W-set is of $(1+\delta)$-dimensional measure zero, $\delta > 0$. The question of whether or not a W-set is necessarily of zero 1-dimensional measure is unanswered except for linear sets, in which case the answer is in the affirmative.

The principal tool used is the integral representation of an analytic function as the potential of a complex mass distribution, the integral being obtained as a limiting form of the Cauchy integral. For example, if E is a compact set of finite 1-dimensional measure which does not divide the plane and if $f(z)$ is regular and bounded near E, then $f(z) = \int_E (\zeta - z)^{-1} d\sigma(z) + g(z)$, where $\sigma(z)$ is a complex mass distribution and g is analytic. The proof of this representation theorem takes advantage of the fact that E can be approximated by an enclosing set of curves whose lengths are uniformly bounded by some $L < +\infty$. Furthermore, the total "mass" of σ, meaning the sum of the total variations of the real and imaginary parts of σ, is bounded by $2ML$, where $|f(z)| \leqq M$. From this it follows that, if E is of zero 1-dimensional measure, the integral term in $f(z)$ is missing, that is, the singularities of f on E are removable. That E is a W-set is an immediate consequence. *J. W. Green.*

Kametani, Syunzi. A remark on the note "On some properties of Hausdorff's measure and the concept of capacity in generalized potentials." Proc. Imp. Acad. Tokyo **20**, 15 (1944). [MF 14867]

It is remarked that a lemma of the first paper reviewed above is unnecessary for the developments of that paper.
M. H. Heins (Providence, R. I.).

Kravtchenko, Julien. Sur la continuité des dérivées du potentiel. J. Math. Pures Appl. (9) **23**, 97–161, 163–210 (1944). [MF 15744]

Various authors have studied the continuity of the derivatives of potential functions using hypotheses including a Hölder condition on the density and frequently a restriction of similar form on the smoothness of a surface. [O. Hölder, Beiträge zu Potenzialtheorie, Diss. Inaug. Stuttgart, 1882; A. Liapounoff, J. Math. Pures Appl. (5) **4**, 241–311 (1898); A. Korn, Ann. Sci. École Norm. Sup. (3) **24**, 9–75 (1907); Abh. Preuss. Akad. Wiss. Phys.-Math.

Kl. 1909, Anhang, no. 2; N. M. Günther, La Théorie du Potentiel, Gauthier-Villars, Paris, 1934.] In chapters I and II of this memoir the author suggests more general problems of this type, proposing to replace Hölder conditions by less restrictive hypotheses. Let $\mu(M)$ be a scalar function of the Cartesian coordinates of M, defined in a bounded domain D of three dimensional space. If, throughout D, $|\mu(M)-\mu(P)| \leqq f(MP)$, where $f(r)$ is a nonnegative nondecreasing continuous function for which $f(0)=0$, then μ is said to have the modulus of continuity $f(r)$. In most of the theory f is subjected to further restrictions of regularity. A modulus $f(r)$ may be replaced by another $f_1(r)$ such that $f_1(r) \geqq f(r)$; hence we may require that $f(r)$ has a derivative when $r>0$ and that $f(r)/r \to \infty$ as $r \to 0$. This excludes $f(r)=cr$, corresponding to a Lipschitz condition. Other requirements corresponding to actual restrictions on μ are that the integral $\varphi(MP)=\int_0^{MP} r^{-1}f(r)dr$ is convergent and that there exists a quantity $K(f, a)$ such that

$$\epsilon \int_\epsilon^a r^{-2}f(r)dr \leqq K(f, a)\varphi(\epsilon), \qquad 0 \leqq \epsilon \leqq a.$$

The continuous function $\varphi(r)$, which is nonnegative and nondecreasing and vanishes with its argument, is called the modulus associated with $f(r)$. The introduction of a suitable constant factor in $f(r)$ assures that $r<f(r) \leqq \varphi(r)$. Any function which has a modulus of continuity f meeting these requirements is said to satisfy a condition (f).

A closed surface S bounding a finite simply-connected domain D of three dimensional space is said to be a surface of Liapounoff if it is made up of a finite number of regular parts $S_1, \cdots, S_n$, such that (1) there is a tangent plane at each ordinary point of S, that is, at each point not on the frontier of an S_i; (2) if (N, N') is the angle between the exterior normals to S at points M and P of S_i then (N, N') satisfies a condition (f), $|(N, N')| \leqq f(MP)$; (3) there exists a constant $R>0$ such that the part of S_i inside any sphere of radius R with center at an ordinary point M of S_i is intersected in at most one point by any line parallel to the normal at M. The $f(MP)$ in (2) is called the modulus of continuity of S. To assure other useful properties the R in (3) is chosen so that $f(R) \leqq 5/13$. Basic characteristics of such surfaces are discussed.

Chapter III is devoted to problems connected with the potential

$$V(M)=\int\int_S \mu(P)\overline{MP}^{-2}\cos(MP, N')d\sigma$$

due to a double layer of moment $\mu(P)$ on a surface of Liapounoff S. It is shown that, if $\mu(P)<A$ (a constant) and $\mu(P)$ is integrable, and if M is an ordinary point of S and M' a point of S such that $MM' \leqq R/8$, then an inequality of the form $|V(M')-V(M)| \leqq k\varphi(4MM')$ is valid. Here k depends on A and on the surface, but is independent of M and M'; φ is the modulus associated with the f of S. This shows that V is continuous on S. Similar inequalities are obtained for $|V(M')-V(M) \pm 2\mu(M)|$, where M is an ordinary point of S and the upper or lower sign is used according as M' is exterior or interior to D. Other formulas relating to the continuity of $V(M)$ are also obtained.

In chapter IV the author discusses the derivatives of the potential

$$U(M)=\int\int_S \overline{MP}^{-1}\mu(P)d\sigma,$$

where S is a surface of Liapounoff. The symbol dU/dn is used to represent

$$\int\int_S \mu(P)\overline{MP}^{-2}\cos(MP, N)d\sigma.$$

On the assumption that $|\mu(P)|<A$ and that M and M' are ordinary points of S such that $MM' \leqq R/8$, an inequality is obtained which may be written in the form

$$\left|\frac{dU(M')}{dn_1}-\frac{dU(M)}{dn}\right| \leqq k'\varphi(4MM') + f(MM')\left\{k''+k'''\log\frac{1}{MM'}\right\}.$$

This assures the continuity of the integral dU/dn at ordinary points of S, provided that $f(r)\log(1/r) \to 0$ as $r \to 0$. [The author claims that the existence of φ implies the desired result; his discussion proves only that the lower limit of $f(r)\log(1/r)$ at $r=0$ is zero.] If $\mu(P)$ satisfies a condition (f) then the interior and exterior normal derivatives exist at ordinary points of S, and are given by the usual formulas in terms of μ and the integral dU/dn. Here and in some later developments results are based on a common modulus $f(r)$ for $\mu(P)$ and S.

Chapter V is concerned with the potential

$$W(M)=\int\int\int_D MP^{-1}\mu(P)d\tau,$$

where D is a finite domain bounded by a closed surface of Liapounoff. If $\mu(P)$ satisfies a condition (f) the second partial derivatives exist at interior points of D. Formulas for these derivatives are obtained.

In chapter VI the author returns to the study of the potential $U(P)$ of chapter IV. If $\mu(P)$ satisfies a condition (f) the tangential derivatives exist at ordinary points of S; formulas are also given for exterior and interior oblique derivatives at such points. Using again the assumption that $f(r)\log(1/r) \to 0$ as $r \to 0$, it is shown that, if M and M_1 are ordinary points of a regular part S_i of S such that $MM_1 \leqq R/16$, then

$$|(\partial U/\partial x)_M-(\partial U/\partial x)_{M_1}| \leqq C\varphi(8MM_1)|\log(1/MM_1)|,$$

where C is a suitably chosen constant. Except for the exclusion of cases in which one of the derivatives is exterior and the other interior, the direction of the x-axis is arbitrary. Other results relating to the continuity of tangential and oblique derivatives are also discussed.

Chapter VII is devoted to the continuity of the second derivatives of the potential $W(M)$ of chapter V. If $\mu(P)$ satisfies a condition (f) and if M and M' are points of D at a sufficient distance from S and sufficiently near each other, then

$$|(\partial^2 W/\partial x^2)_{M'}-(\partial^2 W/\partial x^2)_M| \leqq C'\varphi(4MM'),$$

even in cases in which the boundary of D is not a surface of Liapounoff. A similar inequality involving

$$\varphi(8MM')|\log(1/MM')|$$

and necessitating restrictions on S is given for neighboring points of $D+S$.

Chapter VIII contains a discussion of some refinements and possible extensions of the theory, including a discussion of the dependence of the third requirement for a surface of Liapounoff upon the second. *F. W. Perkins.*

Ivanzov, G. P. Résolution du problème de refroidissement d'un semi-espace rempli d'une matière dont les propriétés sont dépendantes de la température. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 175–176 (1945). [MF 16408]

Let the thermal conductivity and the specific heat of a material be functions of the temperature. The heat equation is then nonlinear. The author considers particular solutions of the equation which are functions of $x/\sqrt{t}$ only, in the case of one-dimensional flow, thus reducing the equation to an ordinary differential equation. He outlines a method of successive approximations to the solutions of the latter equation. *R. V. Churchill* (Ann Arbor, Mich.).

Ivantzov, G. P. Solution du problème de refroidissement d'un semi-espace occupé par une matière à n points critiques dont les intervalles sont caractérisés par des propriétés physiques différentes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 248–249 (1945).

Parodi, Hippolyte. Solution du problème du mur plan indéfini soumis, sur ses deux faces, à des températures périodiques. C. R. Acad. Sci. Paris **222**, 486–488 (1946). [MF 16029]

A vector method is devised for determining the periodic temperatures $\theta(x, t)$ in a wall whose faces are subjected to simple periodic thermal conditions. *R. V. Churchill.*

Karimov, Dsh. H. On the periodical solutions of nonlinear equations of the fourth order. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 618–621 (1945).

P. V. Solovieff [same C. R. (N.S.) **25**, 731–734 (1939); these Rev. **1**, 315] gave the solution of the following periodic boundary value problem:

$$\frac{\partial^2 z(x,t)}{\partial t^2}+\frac{(2k+1)^2}{\pi^2}\frac{\partial^4 z}{\partial x^4}=\Phi(x,t)+\mu f(z), \qquad 0\leqq x, t\leqq 1,$$

$$z(0,t)=z_{xx}(0,t)=z_{xx}(1,t)=0, \quad z(x,0)=z(x,1),$$
$$z_t(x,0)=z_t(x,1).$$

The present paper treats the same type of problem with the function $f(z)$ replaced by $f(z, \partial^2 z/\partial t^2)$. The method of successive approximations is used and the uniqueness of the solution is proved for μ sufficiently small. *F. G. Dressel.*

Bremekamp, H. On the existence of a solution of $\Delta^k u=0$ which together with its $k-1$ first normal derivatives takes given values at the points of a given closed curve. I. Nederl. Akad. Wetensch., Proc. **49**, 185–193 = Indagationes Math. **8**, 82–90 (1946). (Dutch) [MF 16572]

In two previous papers [same Proc. **48**, 222–226, 229–236 = Indagationes Math. **7**, 27–33, 34–41 (1945); these Rev. **7**, 449] the author has proved that the solution of the equation of the title is unique if it has continuous partial derivatives to the order $2k$, inclusive, inside the region under discussion, while its derivatives normal to the boundary up to the order $k-1$ approach the prescribed values if the point under consideration is moved up to the boundary from any direction. Furthermore, special properties of the solution were derived. In the present paper it is proposed to prove the existence of a solution having the same properties, assuming a region bounded by a closed curve having continuous tangents such that the enclosed region may be mapped on the upper half of a plane by conformal representation. The discussion starts from $k=2$. The values of the solution at the boundary are shown to satisfy a system of linear inhomogeneous integral equations of the second kind, which are shown to have a unique solution. When the boundary values are thus determined, internal values may be obtained by the application of Green's theorem.

M. J. O. Strutt (Eindhoven).

Bremekamp, H. On the existence of a solution of $\Delta^k u=0$ which together with its $k-1$ first normal derivatives takes given values at the points of a given closed curve. II. Nederl. Akad. Wetensch., Proc. **49**, 302–311 = Indagationes Math. **8**, 171–180 (1946). (Dutch) [MF 16580]

The author proceeds [see the preceding review] to deal with the case $k=3$. Writing the solution as a sum of even powers of the distance from an arbitrary point inside the region, up to the fourth, the coefficients are shown to be uniquely determined by linear inhomogeneous integral equations of known solution. By application of Green's theorem, the solution inside the region is then uniquely determined. To complete the proof, it is then shown that the proposition holds for order $n+1$ if it holds for n. This proof is also carried out with the aid of Green's theorem.

M. J. O. Strutt (Eindhoven).

Bremekamp, H. On the existence of a solution of $\Delta^k u=0$ which together with its $k-1$ first normal derivatives takes given values at the points of a given closed curve. III. Nederl. Akad. Wetensch., Proc. **49**, 312–318 = Indagationes Math. **8**, 181–187 (1946). (Dutch) [MF 16581]

Continuing his argument [see the preceding review] the author carries out the proof for even values of n. A representation of the solution as a power series similar to that in the preceding case is used. By the application of Green's theorem a similar integral equation for the coefficients of this representation is derived along the boundary curve. It is shown that a uniquely solvable system of equations for the coefficients in the solution is arrived at. As the solution of the problem is known to be unique, the author thus obtains the existence of the solution. *M. J. O. Strutt.*

Bremekamp, H. On the solutions of the equation $\Delta\Delta u=0$ which satisfy certain boundary conditions. Nederl. Akad. Wetensch., Proc. **49**, 319–330 = Indagationes Math. **8**, 188–199 (1946). (Dutch) [MF 16582]

It is proposed to obtain expressions for the solution of the above problem similar to the well-known solutions of Laplace's equation expressed by an integral taken along the boundary curve of the region under consideration, the integrand containing the given boundary values. To effect this, a suitable Green's function is constructed with the aid of Green's theorem. This procedure is then carried out explicitly for a circular boundary curve. The result is similar to Poisson's integral expression for a circle. Subsequently it is proved that the solution thus obtained satisfies the required conditions. The preceding argument pertaining to two dimensions is then extended to three dimensions and a similar result is arrived at. *M. J. O. Strutt* (Eindhoven).

Calculus of Variations

Wagner, V. Geometry of field of local curves in X_3 and the simplest case of Lagrange's problem in the calculus of variations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 229–232 (1945). [MF 16659]

The author considers the problem of Lagrange of mini-

mizing the integral

$$s=\int_{u_1}^{u_2} F(\xi^\alpha, \xi^\alpha')du, \qquad \alpha=1, 2, 3,$$

under the condition $f(\xi^\alpha, \xi^\alpha')=0$, where F and f are positively homogeneous functions of the first degree in the variables ξ^α and $F>0$. For each ξ^α, he calls the curves, defined in x-space by the equation $F(\xi^\alpha, x^\alpha)=1$, $f(\xi^\alpha, x^\alpha)=0$, the indicatrices of the problem of Lagrange, and shows that if the indicatrices (field of local curves) are given then it is possible to write the differential equations of the extremals without using the Lagrange multiplier. In order to do this he assumes that the curves of the field are not plane curves whose planes contain the origin. *J. E. Wilkins, Jr.*

Wagner, V. Geometry of field of local central plane curves in X_3. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 382–384 (1945). [MF 16650]

The author extends the result of the paper reviewed above to the case when the local curves are plane curves whose planes pass through the origin. *J. E. Wilkins, Jr.*

Baiada, Emilio. Un problema non regolare del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **11**, 65–78 (1942). [MF 16753]

The author treats the problem of the existence of a minimum for $I(C)=\int F(x, y, x', y')ds$ when it is not a regular problem by the use of a related integral $J(C)$ which is quasi-regular. *L. M. Graves* (Chicago, Ill.).

Faedo, Sandro. Proprietà asintotiche delle estremanti degli integrali a campo di integrazione illimitato. Ann. Scuola Norm. Super. Pisa (2) **11**, 119–131 (1942). [MF 16757]

The author shows by means of the example

$$I_\infty=\int_0^\infty (y^2+y'^2+2y')dx$$

that a minimizing extremal for an integral over an infinite interval need not satisfy a transversality condition at ∞. In this example the requirement that the integral is finite implies that $\lim_{x\to\infty} y(x)=0$. The author remarks that this example does not fall under previously published existence theorems. *L. M. Graves* (Chicago, Ill.).

Faedo, Sandro. Sull'estremo assoluto degli integrali estesi a un campo illimitato. Ann. Scuola Norm. Super. Pisa (2) **11**, 223–234 (1942). [MF 16763]

The author proves some existence theorems for the absolute minima of integrals extended over an infinite interval, where the integrands are of second degree in the functions $y(x)$ and their derivatives but do not satisfy the hypotheses of previously published existence theorems. *L. M. Graves.*

Faedo, Sandro. Su un teorema di esistenza dell'estremo assoluto in campi illimitati. Ann. Scuola Norm. Super. Pisa (2) **12**, 1–15 (1943). [MF 16765]

Existence theorems proved by Nagumo [Jap. J. Math. **6**, 173–182 (1929)] and by McShane [same Ann. (2) **3**, 287–315 (1934), in particular, p. 298] were extended by Cinquini to the case of infinite intervals of integration and to the case when the integrand involves derivatives of higher order [same Ann. (2) **5**, 169–190 (1936); **6**, 191–209 (1937)]. In the present note the author discusses various modifications of the hypotheses used by the authors cited, and the relations between them. *L. M. Graves* (Chicago, Ill.).

Faedo, Sandro. Un nuovo teorema di esistenza dell'estremo assoluto per gli integrali su un intervallo infinito. Boll. Un. Mat. Ital. (2) **5**, 145–150 (1943). [MF 16100]

Cinquini [Ann. Scuola Norm. Super. Pisa (2) **9**, 253–261 (1940); these Rev. **3**, 248] gave theorems on the existence of a minimum for an integral of the calculus of variations extended over an infinite interval. The present author has extended these results [see, for example, the preceding review]. In this note it is shown that one hypothesis previously used may be dropped altogether, and another hypothesis may be replaced by the equivalent and simpler one that $\lim_{y'\to\infty} f(x, y, y')=\infty$ at each point (x, y). It is remarked that the methods can be extended to the case when more dependent variables and derivatives of higher order are present in the integrand f. *L. M. Graves.*

Faedo, Sandro. Condizioni necessarie per la semicontinuità di un nuovo tipo di funzionali. Ann. Mat. Pura Appl. (4) **23**, 69–121 (1944). [MF 16612]

The author is concerned with the semicontinuity of the two functionals

$$I(y)=\int_a^b\int_a^b f\{x, z, y(x), y(z), y'(x), y'(z)\}dxdz,$$

$$I(y_1, y_2)=\int_a^b\int_c^d f\{x, z, y_1(x), y_2(z), y_1'(x), y_2'(z)\}dxdz.$$

The problem of minimizing the former integral was proposed by Fubini in 1913. The author defines an ordinary curve $c: y=y_1(x)$, $y=y_2(z)$ with $a\leqq x\leqq b$, $c\leqq z\leqq d$ as one for which (1) (x, z, y_1, y_2) is in a region in which f and its first and second derivatives with respect to y_1', y_2' are continuous, (2) $y_1(x)$, $y_2(z)$ are absolutely continuous and (3) $I(y_1, y_2)$ is a finite Lebesgue integral. He then defines the ρ-neighborhood of an ordinary curve c and in terms of this notion defines semicontinuity. It is shown that a necessary condition for the lower semicontinuity of $I(y_1, y_2)$ is that $f_{y_1'y_1'}(x, z, y_1, y_2, y_1', y_2')\geqq 0$ and $f_{y_2'y_2'}(x, z, y_1, y_2, y_1', y_2')\geqq 0$. A necessary and sufficient condition for this lower semicontinuity is that the E-functions of Weierstrass,

$$E_1(x, z, y_1, y_2, y_2'; y_1', \bar{y}_1'), \qquad E_2(x, z, y_1, y_2, y_1'; y_2', \bar{y}_2'),$$

are nonnegative, where E_1 and E_2 are defined in the obvious manner. The author shows that the semicontinuity of $I(y_1, y_2)$ on a curve c cannot be deduced from this property on subarcs of c. In his second chapter he investigates the continuity properties of the functional $I(y)$. The paper contains many results on the continuity of the two functionals on given curves, on subarcs of curves and in an entire region. *H. H. Goldstine* (Princeton, N. J.).

Lewis, J. V. The existence of solutions to Lagrange problems for multiple integrals. Duke Math. J. **12**, 705–722 (1945). [MF 15513]

The author applies two different methods, the second of which is an extension to multiple integral problems of a method developed by E. J. McShane [same J. **2**, 597–616 (1936)], to prove the existence of a vector function $z(x)[=z_1(x_1, \cdots, x_N), \cdots, z_m(x_1, \cdots, x_N)]$ of class $\mathfrak{P}_1$ [see, for instance, J. W. Calkin, same J. **6**, 170–186 (1940); these Rev. **1**, 208; C. B. Morrey, Jr., same J. **6**, 187–215 (1940); these Rev. **1**, 209] which minimizes a Lebesgue integral $I(z)=\int_a f(x, z, p)dx$, $p=D_x z_j$, among all functions $z(x)$ which have boundary values in a given family Γ and which

satisfy a set of differential equations

$$\theta_k(x, z, p) = A^{\alpha}_{k\beta}(x, z)D_{x^\alpha}z_\beta + B_k(x, z) = 0, \qquad k = 1, \cdots, r,$$

almost everywhere. The hypotheses concerning the family Γ are the same for both methods but those concerning f and θ_k are slightly different. For the method of McShane it is assumed that the $A^{\alpha}_{k\beta}$ and B_k are continuous in (x, z) for all (x, z) with x in G and, if S denotes the set in (x, z, p) space where the θ_k are all zero, we have: (1) f is lower semi-continuous on S, (2) there is a convex function $\Phi(p)$ with $\lim_{|p|\to\infty} |p|^{-1}\Phi(p) = +\infty$ such that $f(x, z, p) \geqq \Phi(p)$ for all (x, z, p) on S and (3) for each (x_0, z_0, p_0) in S and each number $k < f(x_0, z_0, p_0)$ there exist numbers $a_\beta{}^\alpha$ and b such that $a_\beta{}^\alpha p^0_{i\alpha} + b > k$ and $f(x_0, z_0, p) \geqq a_\beta{}^\alpha p_\alpha{}^\beta + b$ for all p such that $(x_0, z_0, p) \in S$. If G is a region "of class K" (class C' is a special case), the family Γ is to satisfy the following: (1) Γ is closed with respect to strong convergence in L_1 on G^* (the boundary of G); (2) there is a set E, open on G^*, such that $\int_E |\varphi| dS$ is uniformly bounded for all φ in Γ; (3) there is an admissible z' with boundary values in Γ for which $I(z')$ is finite. Analogous boundary value conditions are set up for the case that G is any bounded region.

C. B. Morrey, Jr. (Berkeley, Calif.).

Fréchet, Maurice. Sur les lignes de discontinuité du plan tangent à une extrémale. Publ. Inst. Mat. Univ. Nac. Litoral 6, 137–144 (1946). [MF 16917]

The author proves the ridge condition for a nonparametric double integral problem in the calculus of variations. He is apparently unfamiliar with earlier discussions. For parametric double integral problems the ridge conditions were, in fact, deduced by Kobb [Acta Math. 17, 321–344 (1893)], while ridge conditions for the general multiple integral nonparametric problem with n dependent variables and m independent variables were formulated by J. E. Powell [Contributions to the Calculus of Variations, 1931–1932, University of Chicago Press, 1933, pp. 445–473] and by A. B. Carson [Contributions to the Calculus of Variations, 1938–1941, University of Chicago Press, 1942, pp. 453–489; these Rev. 4, 48]. *J. E. Wilkins, Jr.*

GEOMETRY

***Courtand, Marc. Sur les courbes gauches du troisième et du quatrième ordre en géométrie finie.** Actualités Sci. Ind., no. 868. Hermann et Cie., Paris, 1940. 85 pp.

Little is known about the closed curves of order n and $n+1$ in projective n-space. The theory of the 3-dimensional special case seems, however, to have reached a point where its presentation appears not only feasible but also desirable. This thesis contains contributions towards the preparation of such a monograph. The curves under consideration are assumed to satisfy certain differentiability conditions. In particular, they have continuous tangents and osculating planes. The order (class, rank) of a curve in 3-space is the maximum number of points of the curve in any plane (of osculating planes through any point, of tangents intersecting any straight line). Order and class of plane curves are defined similarly.

Let Γ_n denote a skew curve of order n. The first part of this paper and an appendix deal with the Γ_3. In chapter I, the plane curves C_3 of third order are reviewed. The author observes that a Γ_3 has class 3 and rank 4 [it may be of interest that the class and rank of any other skew curve are higher]. In the next two chapters, the plane intersections of the surface generated by the tangents of a Γ_3 are described and the following generalization of a theorem by Joachimsthal and Cremona on skew algebraic curves of the third degree is proved [J. Reine Angew. Math. 56, 44–45 (1859); 58, 138–151 (1861), in particular, pp. 146 ff.]. A plane which meets a Γ_3 in three points contains no straight line that is the intersection of two osculating planes. If a plane meets the Γ_3 in only one point, then it contains exactly one such straight line. [The reviewer would like to suggest the following proof of these results. The projection of a Γ_3 is a C_3 of class 3 or 4. A duality transforms a Γ_3 again into a Γ_3 and the C_3 into a curve of order 3 or 4 and class 3; the inflection points go over into cusp points of the first kind and vice versa; the plane intersections discussed by the author correspond to projections, and the generalization of the Joachimsthal-Cremona theorem is equivalent to the fact that a C_3 without a cusp point contains either one inflection point and a double point or three inflection points and no double point [cf. chapter I]. Thus these results would appear as corollaries of the quoted duality theorem [cf. Scherk, Časopis Mat. Fys. 66, 172–191 (1937)].] In chapter IV, the Γ_3's on hyperboloids or convex cones are discussed, while in chapter V and in the appendix special Γ_3's are constructed: a Γ_3 of the algebraic order 7, Γ_3's that are intersections of a convex cone with a hyperboloid or of two such cones, etc. In the last chapter of the first part, the problem of imbedding a skew arc of order 3 in a Γ_3 is solved. It should be noted that Haupt's solution [J. Reine Angew. Math. 170, 154–167 (1934)] makes no differentiability assumptions.

At the beginning of the second part, the author proves that a Γ_4 has only a finite number of singular points. This lemma is a corollary of Haupt's theorem that any arc of order 4 in 3-space, whether differentiable or not, can be decomposed into a bounded number of subarcs of order 3 [Math. Ann. 108, 126–142 (1933)]. In the second chapter, the continuous transformation of the Γ_4 into itself is introduced which maps each point of the Γ_4 onto the point at which its osculating plane intersects the curve again. This mapping permits a characterization of the subarcs of order 3 on the Γ_4. It also enables the author to classify the Γ_4's in the four following chapters according to the number and kind of their singularities. This classification was anticipated by Linsman [C. R. Acad. Sci. Paris 204, 463–465 (1937)]. But this paper also gives the minimum number of arcs of order 3 into which each type can be decomposed. Unfortunately, the author overlooks one type of singularities, the so-called stationary tangents, and thus loses three types of Γ_4's; some details in the first two chapters have to be reformulated [cf., for example, Linsman's note quoted above]. In the last chapter, the following remarkable theorem is proved. If the Γ_4 has no double point and if all of its points lie on trisecants [second type of the author's classification], then these trisecants generate a hyperboloid.

P. Scherk (Saskatoon, Sask.).

***Hjelmslev, Johannes. Examples of geometrical investigations on integral curves in space.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 66–71. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Using various geometrical tools the author finds elegant

solutions of the following problems: to find all curves whose tangents intersect a given plane curve; to find all curves whose tangents are also tangent to a given surface; to determine all integral curves in a null system. *W. Feller.*

Beránek, Jiří. The absolute minimum for rays reflected from plane curves, especially conic sections. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1941, 11 pp. (1941). (Czech. German summary) [MF 16120]

Associated with a plane curve z and a point S in its plane there are three curves p, the caustic c and q such that a light-ray issuing from S, meeting the mirror z at M and being reflected there first meets p at P, then touches c at C, and finally meets q at Q. If a point A on the reflected ray lies between M and P then $SM+MA$ is an absolute minimum (as compared with $SM'+M'A$ where M' is any point on z); if A is between P and C then $SM+MA$ is only a relative minimum and becomes a relative maximum if A is between C and Q; if A is past Q, then $SM+MA$ is an absolute maximum. The author gives a geometrical construction of p and shows that if z is a conic then p is the arc of a confocal conic which passes through S. This follows from the geometrical construction and can also be proved from a theorem of Chasles. *A. Erdélyi* (Edinburgh).

Venkataraman, M. Chain theorems in geometry. Chains of the Cox-Grace type. J. Indian Math. Soc. (N.S.) 9, 1–28 (1945). [MF 15922]

The author proves the following basic proposition. If a chain theorem follows an incidence scheme of the Cox type, the validity of such a theorem is established if it can be shown that the theorems involved in four consecutive links in the chain are true. This proposition is used to prove a number of known chain theorems, as well as to obtain new examples of such theorems. [The basic theorem is substantially the same as that of B. Gambier, Ann. Sci. Ecole Norm. Sup. (3) 61, 199–230 (1944); these Rev. 7, 389.] *N. A. Court* (Norman, Okla.).

Brown, L. M. Some remarks on de Longchamps chain. Proc. Edinburgh Math. Soc. (2) 7, 138–143 (1946). [MF 16209]

Given four circles $S(A)$, $S(B)$, $S(C)$, $S(D)$ all passing through a point P, then any two, say $S(A)$ and $S(B)$, cut again in a point (AB); the three points (AB), (AC), (AD) lie on the circle $S(A)$ and the three points (AB), (AC), (BC) determine a circle $S(ABC)$; then $S(ABC)$, $S(ABD)$, $S(ACD)$, $S(BCD)$ all pass through a point $P(ABCD)$. Let $A, B, \cdots$; (ABC), (ABD), $\cdots$ be the centers of $S(A)$, $S(B)$, $\cdots$, $S(ABC)$, $S(ABD)$, $\cdots$. The author shows that (ABC), (ABD), (ACD), (BCD) lie on a circle if the points A, B, C, D lie on a circle and proves analogous theorems for a "circular complete n-point." Among other contributions to the theory of de Longchamps' configuration is an elementary proof of a theorem of Bath [Proc. Cambridge Philos. Soc. 35, 518–519 (1939); these Rev. 1, 80]. *O. Bottema* (Delft).

Mächler, W. Sequences of cycles on a sphere. Revista Unión Mat. Argentina 9, 171–172 (1943). (Spanish) [MF 16817]

The paper deals with an elementary geometrical theorem concerning sets of mutually tangent circles on a sphere. Vector methods are used. *I. Opatowski.*

Gandini, Adriano. Una notevole applicazione del teorema di Tolomeo. Period. Mat. (4) 23, 85–111 (1943).

Vogliamo applicare il teorema di Tolomeo alla determinazione dell'equazione che lega il lato di un poligono regolare convesso di un numero qualunque di lati al raggio del circolo ad esso circoscritto. *Extract from the paper.*

Convex Domains, Extremal Problems

Fejes, László. Einige Extremaleigenschaften des Kreisbogens bezüglich der Annäherung durch Polygone. Acta Univ. Szeged. Sect. Sci. Math. 10, 164–173 (1943). [MF 16743]

Let K and L be two closed convex curves and $\eta(K, L)$ a distance or deviation between K and L defined in one of the following three ways: (1) η is the least number which is not less than all distances between a point of K and the curve L, and conversely (segment deviation); (2) η is the difference between the areas of the union and the intersection of the regions bounded by K and L (area deviation); (3) η is the difference between the arc lengths of the union and the intersection (length deviation). Now let K be a given convex curve and P_n a polygon with at most n sides for which $\eta(K, P_n)$ is minimal. The author defines $\limsup_{n\to\infty} n^2\eta(K, P_n)$ as the approximation measure of K (with respect to η) and proves that among all convex curves K with given length the circle has the maximal approximation measure with respect to each of the three deviations. The same is valid if only inscribed or only circumscribed polygons are considered. More generally, the closed curves may be replaced by convex arcs with given length and total curvature. A similar result concerning the area deviation and curves with given area has been obtained by the author in a former paper [Ann. Scuola Norm. Super. Pisa (2) 9, 143–145 (1940); these Rev. 3, 90]. *W. Fenchel.*

v. Sz. Nagy, Gyula (Julius). Über konvexe Kurven und einschliessende Kreisringe. Acta Univ. Szeged. Sect. Sci. Math. 10, 174–184 (1943). [MF 16744]

Bonnesen and Kritikos have investigated the annular region bounded by two concentric circles which contains a given closed convex curve and whose radii have minimal difference [for references, see T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, p. 55]. The author considers the annular region with minimal area containing the curve. This region is not always unique; but if it is, the two regions are proved to be identical. Special attention is paid to the case where the second region is not unique. Then the centers of the minimal regions form a segment, and that with maximal inner circle is identical with the minimal region of Bonnesen. Analogous results concerning convex surfaces are stated and, in a footnote, some arguments of Kritikos are simplified. *W. Fenchel.*

Schmidt, Erhard. Über eine neue Methode zur Behandlung einer Klasse isoperimetrischer Aufgaben im Grossen. Math. Z. 47, 489–642 (1942). [MF 15906]

The author gives a unified treatment of isoperimetric problems on two-dimensional manifolds, which includes the results previously obtained [Math. Z. 44, 689–708 (1939); 46, 204–230, 743–794 (1940); these Rev. 2, 12, 262]. The reduction of the n-dimensional case to the two-dimensional one by a "rounding" process is taken up by the author in a

later paper [Math. Z. 49, 1–109 (1943); these Rev. 5, 106]. Let C be a curve enclosing a region T on a 2-dimensional Riemannian space S with line element $ds^2=du^2+g(u)^2dv^2$, $g>0$. For given positive functions $\sigma(u)$ and $\tau(u)$ the integrals

$$L=\int_C \sigma(u)ds, \quad J=\int_T \tau(u)g(u)dudv$$

are considered (L and J reduce to length and area for $\sigma=\tau=1$). The following modified isoperimetric problem is considered first: to find for given L, a, b the curve C with greatest J, which is inscribed in the parallel strip $a\leqq u\leqq b$ and has the given L. It is shown that for L, a, b satisfying

$$(1) \qquad a<b, \quad L>2\int_a^b \sigma(u)du$$

there exists and is uniquely determined (except for translations in the v-direction) an extremal curve C of this problem. The corresponding maximum value of J is a function $G(L, a, b)$ defined in (1). An explicit expression for this function in terms of g, σ, τ is

$$G=\min_{q,k} qL-2\int_a^b \{q^2-\psi(u, k)^2\}^{\frac{1}{2}}\sigma(u)du,$$

where

$$\psi(u, k)=\left[k+\int^u \tau(w)g(w)dw\right]\Big/\sigma(u)g(u)$$

and q and k are arbitrary parameters subject to

$$q>\max_{a\leqq u\leqq b}\psi(u, k).$$

The classical problem of finding for given L the curve C with greatest J (with no prescribed tangential strip) is solved by the extremals of the previous problem that correspond to those values of a, b giving the maximum value of $G(L, a, b)$ for the given L.

Among the special applications may be mentioned the case where S is of constant Gaussian curvature and the curves C are inscribed in a strip bounded by two equidistant curves of constant geodesic curvature, and the problem of determining the "ring-shaped" surfaces of revolution in Euclidean or hyperbolic 3-space of given surface area and enclosing the greatest volume. *F. John.*

Dinghas, Alexander. Verschärfung der isoperimetrischen Ungleichung für konvexe Körper mit Ecken. Math. Z. 47, 669–675 (1942). [MF 15910]

Let K be a convex body in two dimensions, F its area, L its perimeter, and ψ_i the corner angles, if there are corners. Bol [Nieuw Arch. Wiskunde 20, 171–175 (1940); these Rev. 1, 158] proved the sharpened isoperimetric inequality

$$4\pi F-L^2\leqq -4F\sum(\tan\tfrac{1}{2}\psi_i-\tfrac{1}{2}\psi_i),$$

the equality occurring only when K is a circular cap region, that is, a circle to which have been added nonintersecting caps, each bounded by an arc of the circle and a pair of tangent lines. The present paper is devoted to an extension of this result to n dimensions. This involves the construction of an n-dimensional spherical cap body K^* associated with K in a manner analogous to the two-dimensional case. If ψ_i are the semi-angles of the conical caps of K^*, V the n-dimensional volume of K, θ the $(n-1)$-dimensional area of K, W_n the volume of the n-dimensional unit sphere, then the extension obtained is

$$nW_n(nV)^{n-1}-\theta^n\leqq -W_{n-1}(nV)^{n-1}\sum\int_0^{\psi_i}\sin^n x\sec^3 x dx,$$

the equality holding only when K is a cap body. *J. W. Green* (Los Angeles, Calif.).

Dinghas, Alexander. Zum isoperimetrischen Problem in Räumen konstanter Krümmung. Math. Z. 47, 677–737 (1942). [MF 15911]

This paper treats essentially the same problems and uses the same methods as the author's later papers [Math. Ann. 118, 636–686 (1943); Math. Z. 49, 734–792 (1944); these Rev. 6, 101; 7, 25]. The n-dimensional space of constant curvature K is taken here in the Weierstrass form

$$x_1^2+x_2^2+\cdots+x_n^2+x_{n+1}^2/K=1/K,$$

instead of being defined by its metric as in the two papers mentioned. *F. John* (New York, N. Y.).

Differential Geometry

Kasner, Edward. The recent theory of the horn angle Scripta Math. 11, 263–267 (1945).

Expository article.

Kasner, Edward, and De Cicco, John. Comparison of union-preserving and contact transformations. Proc. Nat. Acad. Sci. U. S. A. 32, 152–156 (1946). [MF 16711]

Union-preserving space transformations have been studied previously by the authors [Bull. Amer. Math. Soc. 50, 98–107 (1944); these Rev. 5, 155]. The present paper is concerned with the class of such transformations associated with directrix equations of the form $\Omega(X, Y, Z, x, y, z)=0$. Each such equation also defines a contact transformation in the classical Lie theory; it is shown how the union-preserving and contact transformations defined by the same directrix equation are related geometrically. An illustrative example is provided by the equation

$$(X-x)^2+(Y-y)^2+(Z-z)^2=a^2.$$

The associated contact equation is the well-known dilation for surfaces and the union-preserving transformation defines a quasi-parallelism for curves. *A. G. Walker.*

MacColl, L. A. Geometrical characterizations of some families of dynamical trajectories. Trans. Amer. Math. Soc. 60, 149–166 (1946).

The author's problem is to characterize the family (F) of trajectories of an electrified particle moving in a static magnetic field of force. Specifically, the trajectories of a field of force are studied where the force vector is proportional to the cross product of the velocity and the magnetic induction which depends on the position of the point alone. The family of trajectories of a particle moving according to the laws of special relativistic dynamics in a static magnetic field turns out to be identical with the family (F).

The author gives a completely characteristic set of four properties for a system of ∞^3 curves in space to be a family (F). Use is made of associated planar families of ∞^3 curves, each of which is characterized by a set of four properties. These planar families are of the type (G) considered by Kasner. The family (F) is of the three dimensional type (G). A characterization is given of the general three dimensional

type (G). This corrects a wrong characterization of this type quoted by the reviewer in earlier papers.

The trajectories of the family (F) are subdivided into ∞^1 natural families of ∞^4 trajectories by holding fixed the kinetic energy of the particle. These families are characterized by two independent properties. In the final part of the paper, the two dimensional aspects of the problem are considered. *J. De Cicco* (New York, N. Y.).

Kerawala, S. M. A note on curves congruent to their evolutes. Proc. Nat. Acad. Sci. India. Sect. A. 11, 23–25 (1941).

Behari, Ram. Some properties of rectilinear congruences. Proc. Nat. Acad. Sci. India. Sect. A. 11, 64–66 (1941).

Behari, Ram. A note on 'normal rectilinear congruences.' Proc. Nat. Acad. Sci. India. Sect. A. 14, 98–100 (1944).

In a previous paper [Proc. Indian Acad. Sci., Sect. A. 12, 205–207 (1940); these Rev. 2, 160] I proved the theorem that the curves on the sphere representing the distributive ruled surfaces through a line of a normal rectilinear congruence are isometric. In this paper I have shown that the same result is true for the principal ruled surfaces also.

Author's summary.

Biran, Lutfi. Extension des notions de développée et de développante et leurs images sur la sphère dualistique. Rev. Fac. Sci. Univ. Istanbul (A) 11, 41–46 (1946). (French. Turkish summary) [MF 16714]

The author develops for ruled surfaces an analogue to the theory of evolute and involute for plane curves. The element corresponding to the center of curvature is the instantaneous axis of rotation of the Blaschke trihedron of the ruled surface (which is formed by the generator, the normal to the generator at the central point, and a third line orthogonal to the other two). Ruled surfaces are represented as curves on the unit sphere in dual space ($x+\epsilon y$, $\epsilon^3=0$). *H. Samelson* (Ann Arbor, Mich.).

Haantjes, J. Hexagonal three-webs. Nederl. Akad. Wetensch., Proc. 49, 75–79 = Indagationes Math. 8, 39–43 (1946). [MF 16566]

The author derives the condition that a 3-web in the plane, given by a cubic differential form $p_{ijk}dy^i dy^j dy^k=0$, $i, j, k=1, 2$, is a hexagonal 3-web. The condition is phrased in terms of a Riemannian metric derived from the tensor p_{ijk}. *H. Samelson* (Ann Arbor, Mich.).

Bell, P. O. A study of the projective differential geometry of surfaces by means of a modified tensor analysis. Trans. Amer. Math. Soc. 60, 22–50 (1946).

The author develops a number of projective differential concepts by the use of a tensor notation and an intrinsic derivative. The theory of envelopes and the conditions that a given net be a conjugate net or the asymptotic net are expressed in these new terms. In terms of two invariant quadratic differential forms ϕ and ϕ_2, the author defines the projective normal, mean and total curvatures of one surface relative to another. A geometric characterization of ϕ_2 is given and a connection between it and the projective linear element is established. A duality principle is formulated and applied to certain families of hypergeodesics.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Wong, Yung-Chow. Contributions to the theory of surfaces in a 4-space of constant curvature. Trans. Amer. Math. Soc. 59, 467–507 (1946). [MF 16468]

Par une méthode inspirée de celle de Graustein [Bull. Amer. Math. Soc. 36, 489–521 (1930)], l'auteur étudie quelques problèmes relatifs aux variétés riemaniennes V_2 localement plongées dans un espace S_4 à courbure constante. Certains de ces problèmes ont été antérieurement étudiés par Coburn [Duke Math. J. 5, 30–38 (1939)]. Les résultats les plus intéressants sont ceux relatifs aux V_2 minima (variétés dont le vecteur courbure normale moyenne est nul), aux R-surfaces [cf. Kommerell, Math. Ann. 60, 548–596 (1905)] et aux V_2 réglées dans S_4. Le mémoire s'achève par la résolution complète du problème suivant, posé par Coburn: étant donnée une V_2 d'un S_3, plonger dans un S_4 une variété 2-dimensionnelle, isométrique à V_2, telle que l'un de ses seconds tenseurs fondamentaux dans S_4 soit identique au second tenseur fondamental de la V_2 donnée dans S_3. *A. Lichnerowicz* (Strasbourg).

Sen, R. N. On parallelism in Riemannian space. II. Bull. Calcutta Math. Soc. 37, 153–159 (1945). [MF 16164]

The author continues work begun in part I [same Bull. 36, 102–107 (1944); these Rev. 6, 188], in which a linear connection is defined by means of an orthogonal ennuple of vectors. Tensors based on the author's linear connection and on the ordinary Christoffel symbols are obtained and relationships between these tensors are derived.

M. Wyman (Edmonton, Alta.).

Walker, A. G. A particular harmonic Riemannian space. J. London Math. Soc. 20, 93–99 (1945). [MF 16701]

Walker, A. G. On completely harmonic spaces. J. London Math. Soc. 20, 159–163 (1945).

These papers answer a question that originally arose sixteen years ago. Ruse [Proc. Edinburgh Math. Soc. (2) 2, 135–139 (1931)] claimed to have obtained a formula for Hadamard's elementary solution of the generalized Laplace equation $\Delta_2 V=0$ associated with a Riemannian V_n, the same formula holding for n odd or even. Later, Copson and Ruse [Proc. Roy. Soc. Edinburgh 60, 117–133 (1940); these Rev. 2, 20] showed that the formula was generally valid only for certain types of V_n, tentatively called completely harmonic, but conjectured that such spaces were all of constant curvature. In the papers under review, Walker shows that the conjecture was false, and that completely harmonic spaces do in fact constitute a new class of Riemannian spaces, less general than Einstein spaces, of which they are a special case, but more general than spaces of constant curvature. In the first paper he obtains the metric of a V_4, and thence of a V_n, of the type called simply harmonic, that is, one for which the elementary solution of $\Delta_2 V=0$ reduces to $1/s^{n-2}$, where s is the geodesic distance between a fixed point (x_0^i) and a variable point (x^i), this being the simplest and most obvious generalization of the elementary solution $1/r$ of the ordinary Laplace equation $\nabla^2 V=0$. In the second paper he obtains other V_4's for which Ruse's formula holds, these being completely harmonic without being simply harmonic. In both papers the spaces obtained are symmetric in Cartan's sense, the covariant derivative of the Riemann tensor being everywhere zero. [For a list of papers on harmonic spaces, see the review of a paper by Lichnerowicz and Walker [C. R. Acad. Sci. Paris 221, 394–396 (1945)], these Rev. 7, 395.]

H. S. Ruse (Leeds).

Galvani, Octave. Sur la réalisation des espaces ponctuels à torsion en géométrie euclidienne. Ann. Sci. École Norm. Sup. (3) 62, 1–92 (1945). [MF 16473]

A surface V in ordinary Euclidean space can be regarded as a Riemannian space, in Cartan's sense, by considering it as made up of infinitely small portions of Euclidean planes (the tangent planes) with a law permitting the mapping of two near tangent planes on the same plane. This mapping is based on the definition of parallelism of Levi-Civita. It is a law of connection which has curvature but no torsion. In this work the author sets himself the problem of finding such an "external" image for spaces with torsion. He does this by replacing the surface V of points by a congruence V of linear elements passing through the points of the surface. In stating the corresponding law of mapping (répérage) he is led to a connection which has both curvature and torsion. He also establishes the converse theorem, that any Euclidean point connection (with or without torsion) in 2-dimensional space is locally realisable by a 2-parameter congruence of linear elements in ordinary space. One particular case, in which the linear elements are all normal to the surface, leads to the classical parallelism of Levi-Civita. Another particular case leads to absolute parallelism. Geometrical interpretations of the curvature and torsion are given in terms of the focal elements of the congruence. In particular, the curvature is numerically equal to the product of the inverses of the focal distances.

Some of the work is extended to n dimensions later in the paper, where the author deals with a generalization of the theorem of Schaefli according to which a Riemannian space of n dimensions can always be regarded (locally) as immersed in a Euclidean space of N dimensions ($N \leqq \frac{1}{2}n(n+1)$). For $n>2$, the congruence of linear elements is replaced by the n-dimensional multiplicity of q-plane elements immersed in E_{n+q}. It is proved that an intrinsic point connection with torsion can be defined in such a manifold and that all Euclidean connections can be so realized. *E. T. Davies.*

Norden, A. Sur la géométrie projective-euclidienne de Weyl. Rec. Math. [Mat. Sbornik] N.S. 18(60), 153–166 (1946). (Russian. French summary) [MF 16682]

The paper considers first some geometric properties of projectively Euclidean spaces in general and then applies these results to projectively Euclidean Weyl spaces $(PW)_2$ of two dimensions ($(PW)_n$, $n>2$, degenerate into Riemann spaces and are therefore not considered). It is shown that the geometry of a $(PW)_2$ can be interpreted on a projective plane if one gives a correspondence between the points of intersection of the tangents to two arbitrary curves Γ_1 and Γ_2 and the line joining their points of contact. From this the author gives the explicit values of the affine connection, in a special coordinate system, and also shows that, if g_{ij} and ω_k are the fundamental tensor and vector of $(PW)_2$ while k is its Gaussian curvature, then $\Delta \arcsin(k/a) = 6a(1-(k/a)^2)^{\frac{1}{2}}$ and $w_i = g_i{}^k \nabla_k \arccos(k/a)$, $g_i{}^k$ being the versor defining vector rotation through a right angle. If Γ_1 and Γ_2 are the branches of a curve of second class, the above geometrical characterization coincides with that of Klein, since in this case $(PW)_2$ becomes a space of constant curvature.

M. S. Knebelman (Pullman, Wash.).

Haimovici, Adolf. Sur la notion de quadrique, dans un espace à connexion projective. Bull. Math. Soc. Roumaine Sci. 46, 91–106 (1944). [MF 16512]

Étant donné un espace à connexion projective S_2, au sens de Cartan, l'auteur considère, comme généralisant les quadriques, les surfaces Q qui jouissent de la propriété suivante: le plan tangent à Q au point courant A_0 correspond à A_0 dans une polarité déterminée (II). Le mémoire est consacré à l'étude, par la méthode du repère mobile, des S_2 qui admettent une famille à un paramètre de quadriques. Il est établi dans ce cas que les asymptotiques des surfaces Q sont aussi des géodésiques de l'espace; il en résulte que pour un S_2 riemannien les quadriques Q coincident avec les surfaces de Slebodzinski. Une étude particulière est relative à la détermination de la connexion et de la polarité dans le cas des espaces sans torsion et dans celui des espaces riemanniens. *A. Lichnerowicz* (Strasbourg).

Strubecker, Karl. Differentialgeometrie des isotropen Raumes. II. Die Flächen konstanter Relativkrümmung $K=rt-s^2$. Math. Z. 47, 743–777 (1942). [MF 15913]

In part I of this series [Akad. Wiss. Wien. S.-B. IIa 150, 1–53 (1941)] the isotropic three dimensional space with the singular metric $ds^2=dx^2+dy^2$ was defined. This paper introduces the theory of surfaces of constant curvature K into this space. They are the solutions of the differential equation $rt-s^2=K$. Chapters I and II contain a recapitulation of geometry and kinematics and of the theory of curves in isotropic space. Chapter III deals with the theory of curvature of surfaces. As in ordinary space there are two principal directions and two principal radii of curvature. The lines of curvature, the curvature and the mean curvature have previously been found by E. Müller using another method [Monatsh. Math. Phys. 31, 3–19 (1921)]. As an example the surfaces of revolution are considered. Chapter IV contains the theory of surfaces of constant curvature. The projections of their asymptotic lines on the (x, y)-plane are webs of Tschebyschewsky. The surfaces can be generated by a (left hand or right hand) generalized translation of a curve with torsion $\pm\sqrt{-K}$ along a curve of torsion $\mp\sqrt{-K}$. In chapter V an analytic representation of the surfaces of constant curvature is given. For $K=-1$ this is the representation of Darboux [Leçons sur la Théorie Générale des Surfaces, vol. 3, Gauthier-Villars, Paris, 1894, pp. 273–274] in the form used by Blaschke [Vorlesungen über Differentialgeometrie, vol. 2: Affine Differentialgeometrie, Springer, Berlin, 1923, p. 216]. This chapter contains a great number of references to the literature. Surfaces of revolution and helicoids are studied in chapter VI. The asymptotic lines of all these surfaces, except the spheres, are helices. The converse is true. There is only one rotational surface with $K=-1$. For $K=+1$ there are only three possible cases: (1) the surface of the evolutes of a catenoid, already investigated by Darboux [op. cit., p. 386], A. von Lilienthal [Math. Ann. 62, 539–567 (1905), in particular, p. 563] and H. Graf and R. Sauer [Math. Z. 44, 362–386 (1938), in particular, p. 374]; (2) the surface applicable to a real paraboloid of revolution; (3) the sphere. Chapter VII deals with ruled surfaces of constant curvature. These are the "uneigentliche geradlinige Affinsphären" of J. Radon [Ber. Verh. Sächs. Ges. Wiss. Leipzig 70, 147–155 (1918), in particular, § 2]. [A continuation of the present paper appeared in Math. Z. 48, 369–427 (1942); these Rev. 5, 109.] *J. A. Schouten* (Epe).

Humbert. Sur la géométrie plane dans l'espace attaché à l'opérateur Δ_2. Ann. Univ. Lyon. Sect. A. (3) 4, 93 (1941).

Abstract of a paper already reviewed [J. Math. Pures Appl. (9) 21, 141–153 (1942); these Rev. 6, 18].

Nath, Brij. A note on the pseudo-reciprocal in the wave-tensor calculus. Proc. Nat. Acad. Sci. India. Sect. A. 12, 97–98 (1942).

RELATIVITY

***de Beauregard, Olivier Costa. La relativité restreinte et la première mécanique broglienne.** Mémor. Sci. Math., no. 103. Gauthier-Villars, Paris, 1944. 71 pp.

This is an exposition of the special theory of relativity. It is written to serve as an introduction to relativistic quantum mechanics. In the first part the author enunciates the assumptions of special relativity and develops the usual consequences of these assumptions in such fields as kinematics, optics and electrodynamics. The last part is devoted to the author's own work in placing certain results of relativistic quantum mechanics on a postulational basis.

M. Wyman (Edmonton, Alta.).

Dirac, P. A. M. Application of quaternions to Lorentz transformations. Proc. Roy. Irish Acad. Sect. A. **50**, 261–270 (1945). [MF 16685]

A group of bilinear quaternion transformations is obtained which is equivalent to the Lorentz group. If $q=q_0+iq_1+jq_2+kq_3$, it is shown that the four quantities

$$\frac{2q_1}{1-|q|^2},\quad \frac{2q_2}{1-|q|^2},\quad \frac{2q_3}{1-|q|^2},\quad \frac{1+|q|^2}{1-|q|^2}$$

transform like the components of a world vector under the quaternion transformation $q^*=(aq\pm\mu a)(-\mu aq\pm a)^{-1}$. In this equation, a is an arbitrary quaternion, μ is a pure imaginary quaternion, and both upper or both lower signs must be taken. Application is made to the relativistic addition of velocities. *A. E. Schild* (Pittsburgh, Pa.).

Mosharrafa, A. M. On a positive definite metric in the special theory of relativity. Proc. Math. Phys. Soc. Egypt 2, 61–71 (1946). [MF 16835]

The author studies a real nonholonomic transformation in the coordinate differentials of Minkowski space, which reduces the line element to a positive definite form. A geometric interpretation is given of the Lorentz transformation in the new pseudo-coordinates. *A. E. Schild.*

Wyman, Max. Schwarzschild interior solution in an isotropic coordinate system. Phys. Rev. (2) **70**, 74–76 (1946).

The gravitational potentials for a spherical mass are usually calculated in a coordinate system with the line element of the form $ds^2=e^\nu dt^2-e^\lambda dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2$. The author calculates these potentials in the "isotropic" coordinate system where the line element has the form

$$ds^2=e^\sigma dt^2-e^\rho(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2).$$

A straightforward calculation shows that just as the transformation $\bar r=(1+m/2r)^2r$, $r\geqq a$ = radius of the sphere takes the isotropic exterior solution into the Schwarzschild exterior solution, so the transformation

$$\bar r=(1+m/2a)^2r/(1+mr^2/2a^3),\qquad 0\leqq r\leqq a,$$

does the same for the interior solution. The transformations piece together continuously at the boundary $r=a$.

L. Infeld (Toronto, Ont.).

Narlikar, V. V., and Karmarkar, K. R. Geodesic form of Schwarzschild's external solution. Nature **157**, 515–516 (1946). [MF 16323]

Spherically symmetric geodesic line-elements are obtained which satisfy Einstein's gravitational equations. A particular case is the Schwarzschild solution in a new coordinate system. *A. E. Schild* (Pittsburgh, Pa.).

Pais, A. The energy momentum tensor in projective relativity theory. Physica **8**, 1137–1160 (1941).

After a detailed survey of the general formalism of projective relativity theory a projective energy momentum tensor is set up for a general Lagrange function. The tensor is symmetrical and satisfies the conservation laws. As an application the energy momentum tensor for Dirac's relativistic wave equation is computed. *P. Weiss.*

McVittie, G. C. The regraduation of clocks in spherically symmetric space-times of general relativity. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 147–155 (1945). [MF 16291]

This paper gives a preliminary examination of the question as to whether general relativity admits clock regraduations other than trivial coordinate transformations, a question which has been suggested by recent kinematical theories of relativity. This examination is restricted to spherically symmetric space-times and shows that any such regraduation sets up a correspondence between two space-times with metrics ds^2 and dS^2, where $ds^2=e^{2\sigma}dS^2$, σ being a function of coordinates; in effect one space-time is replaced by the other as a consequence of the regraduation. Applying the field equations of general relativity, it is assumed that units can be chosen so that the numerical values of the constants c, κ, λ remain unaltered by the regraduation; it is also assumed that $t_{ij}=T_{ij}$, where t_{ij}, T_{ij} are the energy tensors of the two space-times. These conditions restrict both σ and the form of ds^2. Only infinitesimal regraduations (small σ) are examined in detail, and it is found that the restrictions on the form of ds^2 are severe. In general, therefore, a space-time does not admit regraduations of the restricted kind considered in this paper. *A. G. Walker.*

Walker, A. G. A theory of regraduation in general relativity. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 164–174 (1946). [MF 15751]

McVittie [see the preceding review] has discussed the problem of temporal regraduations in general relativity, with special reference to spherically symmetrical space-times. The problem is of considerable interest because general relativity has hitherto possessed no theory of time-keeping comparable in profundity with that developed by the Milne school. The paper under review contains an exposition of the problem as it appears in relation to general relativity. It opens with an account of the kinematical properties of space-time and then describes the significance of the energy tensor T^{ij} to an observer having the mean motion of the neighbouring matter. It is noted that this observer's proper-vector h^i (tangent to his world-line) is the unit time-like eigenvector of T^{ij}. Observers having as world-lines the curves defined by the vector-field h^i are called fundamental observers. Next, the author discusses the field-equations

$$(*)\qquad -\kappa T_{ij}=G_{ij}-\tfrac{1}{2}Gg_{ij}+\lambda g_{ij}.$$

These are usually interpreted as defining T_{ij} when g_{ij}, κ, λ are given, or, inversely, as providing a means of determining g_{ij} when T_{ij}, κ, λ are given. The author gives reasons for thinking both interpretations illogical, and concludes that the equations are of the nature of a test: given the totality of local measures of a physical system, then g_{ij} and

T_{ij} are everywhere determined, and the system possesses the structure of general relativity if (*) is satisfied for some constants κ, λ. Thus κ, λ are determined by the field-equations rather than put into them. The field-equations also serve to determine preferential scales of time, length and mass for any particular system. When the field-equations permit changes of scales, the system is said to admit a regraduation. Since κ and λ are dimensional, they may alter in value with a regraduation.

Mathematically, a regraduation is first defined locally for each observer, and it is deduced that, if a regraduation takes place at every event, the old and new tensor-fields, related to the same basic coordinate-system x^i, must be connected by equations of the form $\bar{g}_{ij}=e^{2\sigma}g_{ij}$, $\bar{T}_{ij}=e^{\psi}T_{ij}$. A physical system, together with a basic coordinate system, tensors g_{ij}, T_{ij}, and constants κ, λ satisfying (*), is called a model (M). Models M and $\bar{M}$ are called equivalent if their tensors satisfy equations of the form (*), in which $\bar{\kappa}$, $\bar{\lambda}$ are not assumed to be related in any definite way to κ, λ. If the geodesic hypothesis is adopted (that geodesics are preserved under regraduations), then σ, ψ are constants and the regraduation is of the relatively trivial type called uniform. Excluding this hypothesis, the author finds conditions that a model should admit nonuniform regraduations and discusses them at some length. Attention is confined to the case when $\psi=\psi(\sigma)$ ($\sigma\neq$constant), which includes many cases of physical interest. He reaches a number of remarkable conclusions. For example, a nonstatic Lemaître model can in general be regraduated to become static. Also, for a given Lemaître model, a regraduation can always be found so that $\bar{\kappa}$, $\bar{\lambda}$ have any chosen values ($\bar{\kappa}>0$), including $\bar{\lambda}=0$: the cosmological constant can be annulled. Lemaître models are the only systems examined in detail, because with few exceptions they are the only ones in which pressure is isotropic and which permit nontrivial regraduations. McVittie's results are special cases of some of those of the present paper. The author concludes with the remark that many problems still remain, including that of finding what physical systems admit regraduations for which ψ is not a function of σ. *H. S. Ruse* (Leeds).

Järnefelt, G. Das Einkörperproblem in dem sich ausdehnenden Raume der Einstein-de Sitter'schen Welt. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 12, 38 pp. (1942). [MF 16154]

The first part of the paper deals with the reduction of certain differential forms

$$ds^2=e^{\nu(r,\,t)}dt^2-e^{\lambda(r,\,t)}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)$$

to the standard coordinates ρ, τ in which

$$ds^2=H(\rho,\,\tau)d\tau^2-G(\rho,\,\tau)d\rho^2-\rho^2d\theta^2-\rho^2\sin^2\theta d\varphi^2.$$

The two cases considered are (i) the Einstein-de Sitter universe in which $\nu=0$, $\lambda=g=\frac{4}{3}\log(1+\frac{3}{2}k_0t)$, $k_0=$constant, and (ii) McVittie's form for a particle of mass m at the origin of coordinates in this universe for which

$$\nu=2\log\left(\frac{1-m/2re^{\frac{1}{2}g}}{1+m/2re^{\frac{1}{2}g}}\right),\qquad \lambda=4\log(1+m/2re^{\frac{1}{2}g})+g.$$

The functions H and G are found in McVittie's case as power series in the two variables $x=\rho^{-1}$, $y=k_0^2\rho^2/(1+\frac{3}{2}k_0\tau)^2$. These series are then generalized and a more general field than McVittie's is obtained, still with a particle of mass m at the origin. In this more general field, the orbits of planets traced out round the particle are shown to be unaffected by the expansion of the universe as a whole. *G. C. McVittie.*

Morris, T. F. The two-body problem in Einstein's and Birkhoff's theories. Phys. Rev. (2) 69, 541 (1946). [MF 16528]

The differential equations found by Graef Fernández [Bol. Soc. Mat. Mexicana 1, nos. 4 and 5, 25–39 (1944); these Rev. 6, 240] for the motion of the centre of gravity in the two-body problem of Birkhoff's theory are compared with those found by Einstein, Infeld and Hoffmann [Ann. of Math. (2) 39, 65–100 (1938)] on the basis of general relativity. The latter contain terms not appearing in the former, and the discrepancy cannot be explained by accepting the suggestion made by Berenda [Phys. Rev. (2) 67, 56 (1945)] that differences in the two theories may be due to different methods of approximation. The extra terms are interaction terms arising from the nonlinearity of the field equations. The Einstein equations give no secular change in the velocity of the centre of gravity, but in the Birkhoff theory there is an acceleration in the direction of the periastron of the minor component of a double star. Levi-Civita [Amer. J. Math. 59, 225–234 (1937)] has given an example of a binary star for which such an acceleration might be detected. *H. S. Ruse* (Leeds).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Synge, J. L. Reflection in a corner formed by three plane mirrors. Quart. Appl. Math. 4, 166–176 (1946). [MF 16962]

When light is reflected at each of three plane mirrors in succession, the resultant ray-transformation is equivalent to a single rigid-body rotation accompanied by a reversal of sense. As the order of reflection at the three mirrors is capable of 6 permutations, a set of 3 mirrors ordinarily gives rise to 6 distinct transformations; these have the same angle of rotation, but different axes. In particular, reversal of the order of reflections results in exact reversal of direction of the axis. Representing directions by points on the surface of a unit sphere, it is shown that the points N_1, N_2, N_3 representing the normals to the mirrors are the bisectors of the sides of a spherical triangle formed by the points A_1, A_2, A_3 which represent the axes of rotation of the three transformations corresponding to a set of cyclic permutations of the order of reflection. Formulae are given for the vectors A_i, which are linear functions of the vectors N_i, and for the angle of rotation, which equals the spherical defect of $\triangle A_1A_2A_3$. Modifications are given for cases in which the mirrors are perpendicular to each other.

A. J. Kavanagh (Buffalo, N. Y.).

Mandelshtam, L. Perfect optical image from the viewpoint of wave optics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 302–305 (1946). (Russian. English summary)

A formalism is set up, under simplifying assumptions, for the transformation by an optical instrument of an arbitrarily given wave-displacement function in the object-plane into the corresponding wave-displacement in the

image-plane. This is connected with the practical problem of the imaging of a transilluminated object under fully coherent illumination. The principal simplifications are the use of a scalar wave-function and the restriction of the discussion to the two-dimensional form of the problem. Perfect imaging is defined as the transformation by the system of an arbitrary displacement-function $f(x, y)$ in the object-plane (coordinates x, y) into a displacement-function of the form $\mu f(\lambda x', \lambda y')$ in the image-plane (coordinates x', y'), where λ, μ are constants.

By the use of Fourier integral transforms, it is shown that perfect imaging between two planes is only possible in telescopic systems satisfying the sine-condition and that perfect imaging is only possible between two volumes in the cases of plane reflection and of identical imaging. There is no attempt at rigour in the mathematical arguments.

E. H. Linfoot (Bristol).

Hopkins, H. H. Monochromatic lens aberration theory. Philos. Mag. (7) **36**, 546–568 (1945). [MF 15951]

Aberrations in a system of spherical refracting surfaces are considered in terms of the departures of refracted waves from a truly spherical shape. Series expansions are given for the departures as functions of the data of triangulated rays. A theory corresponding to the Seidel theory of ray-optics is developed. Application is made to the problem of best focus in the presence of aberration. A concept of paraxial image surface is developed for fields of finite size; this surface is not in general the locus of foci of elemental portions of the several waves actually emerging from the system unless these waves are free of wave-front aberration.

A. J. Kavanagh (Buffalo, N. Y.).

Rabinovich, G. D. Investigation of aberrations of optical systems on the basis of ikonal theory. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 161–170 (1946). (Russian. English summary)

In the first section of the paper it is shown that Schwarzschild's Seidel ikonal can be generalised without difficulty to the case of an optical system without any axis of symmetry. The remainder of the paper is concerned with systems possessing an axis of symmetry and applies an ikonal expansion to discuss their off-axis aberrations. A new classification of these aberrations is proposed and methods are outlined for their approximate determination in a practical case by trigonometrical ray-tracing involving only meridional rays.

E. H. Linfoot (Bristol).

Kousnetsov, E. S. Application of the formulae of the theory of non-horizontal visibility to the calculation of the sky's brightness and the visual range for the simplest forms of the indicatrix of scattering. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 204–229 (1945). (Russian. English summary) [MF 15351]

Certain Fredholm integral equations of the second kind arising in the theory of the scattering of light in the atmosphere are solved numerically by the method of successive approximations for special values of the parameters appearing in them, and the results are applied to the calculation of the distribution of sky brightness and the range of visibility of objects against the sky and against the earth's surface. In the calculation of sky brightness both spherical scattering and a simple case of nonspherical scattering are considered; reflection of light at the earth's surface is taken into account throughout.

F. Smithies.

Davy, N. The two-dimensional electric field of a single semi-infinite rectangular conductor. Philos. Mag. (7) **36**, 694–705 (1945). [MF 16976]

The electric fields of two two-dimensional conductors are investigated theoretically, and some of the conclusions are checked experimentally. The conductors are (a) a single semi-infinite rectangular cylinder charged externally, and (b) a single semi-infinite thin-walled hollow rectangular cylinder charged internally. *From the author's summary.*

Belluizi, A. L'induzione elettromagnetica in un suolo omogeneo isotropo d'una bobina circolare in un piano verticale. Rend. Sem. Fac. Sci. Univ. Cagliari **15**, 49–59 (1945). [MF 16631]

Corben, H. C. A classical theory of electromagnetism and gravitation. I. Special theory. Phys. Rev. (2) **69**, 225–234 (1946). [MF 15824]

The author's aim is to unify classical electrodynamics with Newtonian gravitational theory. A 5-dimensional flat space is introduced by the addition of a fifth dimension (expressed by a time-like coordinate t') to the x, y, z, t of special relativity. A set of equations analogous in form to Maxwell's equations is written down which contain, in addition to the electromagnetic vectors **E** and **H**, a vector **F** which is the gravitational field-strength and scalars $c\theta$, σ_0, $\mathfrak{W}$ which are, respectively, the gravitational rate of work on unit mass, the mass density and the total rate of work. If physical quantities do not involve t', a theory invariant for Lorentz transformations results in which the gravitational potential is a Lorentz invariant, gravitational waves are propagated with the velocity of light and gravitational force acts on the rest mass of a particle. The conservation laws are worked out. An accelerated mass emits energy in the form of gravitational waves and has a gravitational self-energy, the classical radius for mass m being Gm/c^2, where G is the gravitational constant.

G. C. McVittie.

Broer, L. J. F. Note on the theory of vector wave fields. Nederl. Akad. Wetensch., Proc. **48**, 190–197 (1945). [MF 16585]

The possibility of considering the electromagnetic field as the limiting case of the vector wave field when $\kappa(=mc/\hbar)\to 0$ is investigated. It is shown that this is possible in the classical but not in the quantum theory.

C. Kikuchi.

Jaeger, J. C. On the solution of the Maxwell equations in regions bounded by perfectly conducting surfaces of the cylindrical coordinate system. J. London Math. Soc. **20**, 154–159 (1945).

The author shows that the methods employed by Carslaw and Jaeger [same J. **15**, 273–281 (1940); these Rev. **2**, 292] in the theory of heat conduction can be taken over with very little modification to give solutions of the time dependent Maxwell equations in the same regions with perfectly conducting boundary surfaces.

A. E. Heins.

Feld, J. N. Initial boundary problems of electrodynamics. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 172–174 (1945). [MF 16664]

The paper gives a formulation and solutions of certain initial boundary problems of electrodynamics of unsteady oscillations. Three cases discussed are those for which (1) the tangential component of the electric vector is given for $0\leqq\tau\leqq t$ on the bounding surface s, (2) the magnetic instead of the electric vector is specified on the boundary and

(3) electric and magnetic vectors are given on portions s_1 and s_2, respectively, of the surface s. The analysis is based on a generalized lemma of Lorentz derived earlier by the author [same C. R. (N.S.) **41**, 280–283 (1943); these Rev. **6**, 221]. *C. Kikuchi* (East Lansing, Mich.).

Feld, J. The boundary problem of electrodynamics and integral equations of some diffraction problems. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **14**, 330–341 (1944). (Russian) [MF 15379]

The paper discusses the "first boundary problem of electrodynamics," namely, the problem of determining the electromagnetic field in a volume v bounded by a closed surface s, on which is given the tangential component of the electric vector; the volume v may contain a known distribution of radiating sources. It is supposed throughout that the fields oscillate harmonically with a known frequency. A general solution is given, in terms of surface and volume integrals, which is shown to satisfy Maxwell's equations inside s and to possess, at "regular" points of s, limiting values which satisfy the prescribed boundary conditions.

The general solution is applied to discuss the problem of diffraction through an opening s_0 in an infinite, perfectly conducting flat screen. The method is to solve the boundary problem separately in each half-space, in terms of the (unknown) distribution E_t of the tangential electric field over the plane opening s_0; the solutions have to satisfy the Sommerfeld radiation principle at infinity. Then the condition that the tangential component of the magnetic vector shall be continuous on passing through s_0 yields a pair of integral equations which determine E_t on s_0; these have the form

$$\int_{(s_0)} E_x G(q', q)ds = f(q'), \quad \int_{(s_0)} E_y G(q', q)ds = F(q'),$$

$q' \epsilon s_0$, where $f(q')$, $F(q')$ are theoretically determinate functions and the Fredholm kernel $G(q', q)$, also a known function, is symmetric in q and q'; q denotes the point at which the element ds is situated.

Two examples are briefly discussed: (1) diffraction through an annular aperture of the radiation from a dipole oscillator placed on one side of a plane conducting screen, the axis of the dipole lying along the normal through the center of the annulus; (2) diffraction of plane-polarised waves through a slit, the magnetic vectors being parallel to the edges of the slit. In the second example, elliptic coordinates are introduced and a solution is obtained in terms of Mathieu and Mathieu-Hankel functions. *E. H. Linfoot.*

Feld, J. N. The general reciprocity theorem in the theory of receiving and transmitting antennae. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 476–478 (1945). [MF 16648]

Let $I^{(1)}$ and $I^{(2)}$ denote the currents passing through the first and second antennas, respectively, if operated as receiving aerials, E_1 and E_2, the emf's of the generators connected to the terminals for transmitter operation; Z_1 and Z_2 are the antenna impedances of the generators, $Z^{(1)}$ and $Z^{(2)}$, the receiver impedances, Z_{a1} and Z_{a2} the lumped antenna impedances across the terminals and

$$E_j^{ext} = E_j - I_j Z_j = I_j Z_{aj},$$

$j=1, 2$, and similarly $E^{(j)ext}$ the associated external potential differences. The reciprocity theorem proved here is stated in the three equivalent forms

$$I^{(1)}E_1^{ext} - I_1 E^{(1)ext} = I^{(2)}E_2^{ext} - I_2 E^{(2)ext},$$
$$I^{(1)}I_1(Z_{a1}+Z^{(1)}) = I^{(2)}I_2(Z_{a2}+Z^{(2)}),$$
$$I^{(2)}E_1(Z_{a2}+Z^{(1)})/(Z_{a1}+Z_1) = I^2E_2(Z_{a2}+Z^{(2)})/(Z_{a1}+Z_1).$$

For the case, usually considered, of equal transmitting and receiving impedances, $Z_j = Z^{(j)}$, the reciprocity theorem reduces to the well-known classical form $I^{(1)}E_1 = I^{(2)}E_2$. On the other hand if, as usually done in the theory of receiving antennae, the concept of the total receiving emf, $E^{(j)} = I^{(j)}(Z_{aj}+Z^{(j)})$, is introduced, the general theorem appears in the compact form $I_1E^{(1)} = I_2E^{(2)}$ which appears to be much like the classical one but is not equivalent to it. *H. G. Baerwald* (Cleveland, Ohio).

Feld, J. N. Diffraction antennae with axial symmetry. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 115–118 (1946).

The diffraction antenna is an infinitely thin perfectly conducting surface of revolution from which a number of narrow belts have been cut out, each bounded by two parallels. The antenna is excited by an oscillating dipole lying along the axis of revolution. The author derives formulas for the fields inside and outside the antenna in terms of auxiliary fields satisfying prescribed conditions across the gaps. The case of a spherical antenna with a dipole at the origin and only one gap is discussed in detail. *M. C. Gray* (New York, N. Y.).

Feld, J. N. Radiating surface systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 203–206 (1946).

A generator inside a closed metal surface, of dimensions comparable with the wavelength, may be coupled to the exterior of the surface by means of linear conductors to provide a radiating surface. The author considers a spherical surface with a long radial coupling element. It is assumed that the current distribution on this element is known and formulas are derived for the current on the spherical surface. *M. C. Gray* (New York, N. Y.).

Niessen, K. F. The ratio between the horizontal and the vertical electric field of a vertical antenna of infinitesimal length situated above a plane earth. Philips Research Rep. **1**, 51–62 (1945). [MF 16413]

The author uses the Sommerfeld formula for the Hertzian vector of a vertical dipole at height z_s above a plane earth to obtain the ratio of the horizontal and vertical fields in the air at small angles of elevation:

$$\left|\frac{E_r}{E_z}\right|^2 = \frac{\sin^2\alpha + A^{-1}(k_1^{-2}z^{-2} - (2n/k_1R)\sin\alpha\sin\chi/2) + k_1z_sB}{\cos^2\alpha(1+k_1z_sC)},$$

where α is the angle of elevation of the point (z, r) at which the fields are received, $\sin\alpha = z/R = z/(r^2+z^2)^{\frac{1}{2}}$. The other parameters are defined as follows:

$$k_1 = \omega/c = 2\pi/\lambda; \quad k_2^2 = (\epsilon_2\omega^2 + i\omega\sigma_2)/c^2, \quad k_2/k_1 = ne^{i\chi/2},$$
$$A = 1 + 2n\cos\chi/2\sin\alpha + n^2\sin^2\alpha, \quad B = 2\sin\chi/2/(nAk_1^2z^2),$$
$$C = 2\cos^2\alpha(1+n\sin\alpha\cos\chi/2)/(Ak_1z).$$

The range of validity of the formula is investigated. In addition to the usual assumptions ($R \gg \lambda$, $|k_2^2| \gg k_1^2$) the restriction $2\pi R/\lambda \gg n/(A\sin\alpha)$ must be imposed. *M. C. Gray* (New York, N. Y.).

Bouwkamp, C. J. Radiation resistance of an antenna with arbitrary current distribution. Philips Research Rep. **1**, 65–76 (1946). [MF 16414]

For a prescribed current distribution function $I(s)e^{i\omega t}$

on an infinitely thin antenna of length $2l$ energized at the base, the radiated energy can be expressed as a repeated integral

$$W=(2/c)\int_0^{2a}K(t)dt\int_0^{2a-t}f(\xi)f(\xi+t)d\xi,$$

where $\xi=2\pi s/\lambda$, $a=2\pi l/\lambda$, $f(\xi)=I(s)$ and

$$K(t)=(\sin t-t\cos t)/\beta.$$

The formula is easily modified to cover the symmetrical case of an antenna over a perfectly conducting earth. The integral is evaluated for various special forms of the function f. *M. C. Gray* (New York, N. Y.).

Bouwkamp, C. J., and de Bruijn, N. G. The problem of optimum antenna current distribution. Philips Research Rep. 1, 135–158 (1946). [MF 16415]

This is a critical discussion of a paper on the same subject by La Paz and Miller [Proc. I.R.E. 31, 214–232 (1943); these Rev. 5, 53]. It is shown that there is no minimizing function in the sense suggested in the earlier paper and also that the proposed minimizing sequence of current distribution functions does not lead to the best available approximation.

The authors discuss a set of functions defined by

$$f_n(a,\xi,\beta)=(\beta/a)(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}\beta^2\xi^2/a^2)\sum_{m=0}^{n}\binom{m}{n}(\beta/a)^{2m}H_{2m-1}(\beta),$$

where H is Hermite's polynomial and β is arbitrary except for the restriction that it must be greater than the largest root of $H_{2n}(u)=0$. Examples are given to show that the parameters can be chosen to realize approximately any given vertical radiation pattern and to give current distributions much superior to the "optimum current distribution" suggested by La Paz and Miller. *M. C. Gray.*

King, Ronold, and Schelkunoff, S. A. Discussion on "Concerning Hallén's integral equation for cylindrical antennas" by S. A. Schelkunoff. Proc. I.R.E. and Waves and Electrons 34, 265–269 (1946). [MF 16301]

King criticizes Schelkunoff's recent paper [Proc. I.R.E. 33, 872–878 (1945); these Rev. 7, 271] and states that his (King's) results are in better agreement with experimental findings than either Schelkunoff's or Gray's [J. Appl. Phys. 15, 61–65 (1944); these Rev. 6, 282]. He supports his contention with two tables in which results of the various theories are compared numerically.

Schelkunoff in his reply points out that his criticism was directed against Hallén's original theory on which King's former (but not the most recent) work was based.

A. Erdélyi (Edinburgh).

King, Ronold, and Middleton, David. Correction and supplement to our paper: The cylindrical antenna: current and impedance. Quart. Appl. Math. 4, 199–200 (1946). [MF 16967]

The paper appeared in the same Quart. 3, 302–335 (1946); these Rev. 7, 401.

Harrison, Charles W., Jr. Calculation of the impedance properties of parasitic antenna arrays involving elements of finite radius. J. Amer. Soc. Naval Engineers 57, 224–239 (1945).

The author uses the Hallén method to obtain formulas for the current distribution in a system of two parallel cylindrical antennas of unequal length and radius. The antennas are assumed to be center-driven and arranged at right angles to the line joining their centers. The input impedances of the antennas are determined in two special cases, when the applied voltages are equal in magnitude and either in phase or in phase opposition. The self and mutual impedances for any other driving conditions are then expressed in terms of these four impedances. The design problem for optimum field conditions when one antenna is parasitic is also discussed. *M. C. Gray.*

Mendez D., Eugenio, Morones C., Luis, and Zorrilla, Manuel. Electromagnetic radiation of plates. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario 1944, 95–99 (1945). (Spanish)

Pande, A. Sky-wave transmission with variable angle of radiation. Proc. Nat. Acad. Sci. India. Sect. A. 12, 66–80 (1942).

Hoffmann, Banesh. Tensors and equivalent circuits. J. Math. Phys. Mass. Inst. Tech. 25, 21–25 (1946). [MF 16147]

In a paper in Trans. Amer. Inst. Elec. Engrs. 62, 25–31 (1943) [these Rev. 4, 151] Kron desires to express his results in the form of tensor equations in order to see how to set up equivalent circuits for his problem, which can then be solved on a network analyzer. The first section of Kron's paper purports to outline his method as the determination of a quantity C making possible the replacement of a nontensor equation $A+B=0$ by a tensor equation

$$(A+C)+(B-C)=0,$$

where A, B, and C are not tensors, but $A+C$ and $B-C$ are. It being clear, however, that zero cannot here be both a tensor and not a tensor, Hoffman was led to ask if Kron's main results are still valid. The body of Kron's paper is indeed correct, and Hoffmann, following the manipulations of the paper in detail, shows that in essence the method followed depends on the fact that a nontensor equation $dV_a=0$ (corresponding to $A+B=0$) will have the same form as the tensor equation $\delta V_a=0$ (covariant differential, corresponding to $(A+C)+(B-C)=0$), by virtue of originally given equations of motion $V_a=0$ (corresponding to $C-C'=0$). The explicit recognition of this also allows some simplification of the derivation. *L. C. Hutchinson.*

Rytov, S. M., Prokhorov, A. M., and Zhabotinsky, M. E. On the theory of the frequency stabilization. I. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 15, 557–572 (1945). (Russian. English summary) [MF 15636]

Rytov, S., Prokhorov, A., and Zhabotinsky, M. On the theory of the frequency stabilization. II. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 15, 613–628 (1945). (Russian. English summary) [MF 15635]

Perturbation methods, following Poincaré and Liapounov, are applied to several circuit problems with nonlinear elements. Stability is considered. Quantitative as well as qualitative checks with experiment are obtained.

N. Levinson (Cambridge, Mass.).

Colombani, Antoine. Les polynomes de Tchebycheff et la théorie des filtres électriques. C. R. Acad. Sci. Paris 222, 1278–1280 (1946). [MF 16734]

The author starts from the well-known difference equations pertaining to an endless filter circuit consisting of equal cells. These equations lead to polynomials in the ratio

of the series and the parallel impedances of the filter. These polynomials, if multiplied by a normalizing factor, are identical with Chebyshev polynomials. The properties of these polynomials permit one to derive the values of the characteristic frequencies of the maxima as well as some properties pertaining to the response curve as dependent on frequency and on the number of cells. *M. J. O. Strutt.*

Parodi, Maurice. Sur l'existence du groupe d'ensembles à n mailles indépendantes ayant les mêmes fréquences propres qu'un ensemble donné à n mailles et dont la somme des paramètres totaux de chaque espèce (self-inductances, élasticités ou résistances) est égale à la somme des paramètres totaux de même espèce de l'ensemble initial. C. R. Acad. Sci. Paris 222, 1166–1167 (1946). [MF 16854]

Quantum Mechanics

Courtois, Jacques. Les expanseurs, cas particuliers et applications. C. R. Acad. Sci. Paris 222, 864–866 (1946). [MF 16285]

The author shows how one can describe certain basic physical quantities by using expansors [cf. the same C. R. 222, 377–378, 480–482 (1946); these Rev. 7, 411]. For instance, expansor descriptions for wave-length, period, phase velocity, and the potential energy of a system of particles are presented. *A. Schwartz* (State College, Pa.).

Ma, S. T. On the Heisenberg picture in quantum electrodynamics. Chinese J. Phys. 6, 36–49 (1945). (English. Chinese summary)

Heisenberg introduced a form of quantum theory of radiation in which the interaction of the electrons and electromagnetic field is treated without making any use of their Hamiltonian and the Schrödinger equations for their representatives. Thus its connection with the original theory developed by Dirac, that of the variation of parameters, is not clear although identical results have been obtained for some specific problems. This paper deals with the proof of the general equivalence of the two theories by means of the transformation theory of quantum mechanics. In the first part of the paper Heisenberg's theory of emission, absorption, and scattering is shown quite generally to lead to the same result as Dirac's. In the second part Weisskopf's extension of Heisenberg's method to eigenvalue problems is presented in a way that shows more clearly its quantum mechanical interpretation, and then its equivalence to the method of the perturbation theory of stationary states is given. *S. Kusaka* (Princeton, N. J.).

Gustafson, Torsten. Elimination of divergencies in quantum electrodynamics and in meson theory. Nature 157, 734 (1946). [MF 16837]

This note points out that M. Riesz's method of analytical continuation is an alternative method to the λ-process of Wentzel and Dirac for the elimination of the divergences which occur in the classical theory of the electron. The author applies the method to quantum theory and shows that the self-energy term of a nucleon due to the scalar meson field corresponding to the electrostatic interaction is finite and has the value $g^2\kappa/8\pi$, while that of the electron due to the electromagnetic field vanishes. There are, however, other terms in the self-energy due to the vacuum field fluctuations which diverge and can only be eliminated by the introduction of Dirac's negative energy hypothesis. *S. Kusaka* (Princeton, N. J.).

Rayski, Georg. An attempt to formulate a divergence-free quantum mechanics of fields. Nature 157, 873 (1946).

Blokhinzev, D. Note on the possible relativistic-invariant generalization of the concept of field. Acad. Sci. USSR. J. Phys. 10, 167–169 (1946).

Flint, H. T. Quantum equations and nuclear field theories. Philos. Mag. (7) 36, 635–643 (1945). [MF 16980]

This is a continuation of the author's previous works on the geometrical formulation of quantum equations. Using Kaluza's five-dimensional scheme of general relativity theory including the electromagnetic phenomena, the author derives the Dirac equation for nuclear particles in the presence of gravitational and electromagnetic fields. Then in order to introduce the meson field which gives rise to the interaction between the nucleons, he incorporates a metrical scheme after the manner of Weyl and derives generalized equations which include the meson interaction terms. Explicit expressions are obtained for different types of meson fields. The author also points out the analogy between the equations and those for the electromagnetic field in polarizable media. Taking the particular case where the mesons are described by a tensor $V_{\mu\nu}$, he shows that, if a tensor $S_{\mu\nu}$ is introduced by means of the relation $V_{\mu\nu}=T_{\mu\nu}+4\pi S_{\mu\nu}$ in analogy with the relation between the displacement, the field, and the polarization tensors, then $S_{\mu\nu}$ is just the expression in the nuclear wave function which generates the meson field. *S. Kusaka* (Princeton, N. J.).

Nørlund, Ib. Undor representation of the five-dimensional meson theory. Danske Vid. Selsk. Math.-Fys. Medd. 19, no. 9, 29 pp. (1942). [MF 15393]

The author first gives a review of the well-known correspondence between a two-index four-component spinor (a quantity which transforms as the product of two solutions of the Dirac equations) and the following set of tensor quantities: a scalar, a four-dimensional vector, a four-dimensional antisymmetric two-index tensor, a four-dimensional antisymmetric three-index tensor (a pseudo-vector) and a four-dimensional four-index antisymmetric tensor (a pseudo-scalar). It is also known that the vector and pseudo-scalar may be considered as the components of a five-dimensional vector, whereas the antisymmetric tensor and the pseudo-vector form an antisymmetric five-dimensional two-index tensor.

The equations satisfied by the functions describing the symmetrical meson theory and the neutral meson theory have been written in five-dimensional form by C. Møller [same Medd. 18, no. 6 (1941); these Rev. 3, 63]. The author uses the correspondence mentioned above to write these equations in terms of two-index spinors. The symmetrical meson theory involves a vector and a pseudo-scalar and since these are the components of a five dimensional vector no difficulty is encountered in generalizing these equations to five dimensions and determining a five-dimensionally invariant Lagrangian function from which the equations may be derived by a variational principle.

The neutral meson theory involves a scalar and a pseudo-vector. Because these are not components of the same geometrical five-dimensional being, it is to be expected that difficulties will arise in the five-dimensional generalization of these equations. The author finds that it is impossible to determine a five-dimensionally invariant Lagrangian function from which the neutral meson theory equations may be derived by a five-dimensional variational principle.

A. H. Taub (Princeton, N. J.).

Flügge, S. Eine Bemerkung zur Theorie des Mesonfeldes. Ann. Physik (5) **43**, 573–577 (1943).

Extending the similarity between the meson field and the electromagnetic field, the energy in the former is investigated in the presence of an extended source function in analogy to the existence of continuous distribution of charge in Maxwell's theory. It is shown that the Hamiltonian of the system can be diagonalized with the proper eigen-values without the usual perturbation approximations if the charge density is an ordinary number, that is, in the case of the neutral scalar meson theory. Calculation of the interaction energy is carried out explicitly for a uniform distribution of mesic charge in a sphere and applied to the binding energy of heavy nuclei. The result is entirely analogous to the Coulomb energy of a homogeneously charged sphere and reduces exactly to that value when the mass of the meson is allowed to vanish. In the opposite limit when the Compton wave length of the meson, that is, the range of the nuclear force, is small compared to the radius of the nucleus, the interaction energy is proportional to the volume of the nucleus and to the square of the mesic charge density. Thus this theory, of course, does not give the observed saturation property of the binding energy, but this result was to be expected since neutral scalar mesons give rise only to ordinary force and to no type of exchange force between nucleons.

S. Kusaka (Princeton, N. J.).

Yamasaki, Zyunpei. On the meson theory involving a mixture of scalar and pseudoscalar field. I. Proc. Phys.-Math. Soc. Japan (3) **25**, 659–666 (1943). [MF 15077]

A theory of the interaction between nucleons and the meson field is formulated in which the coordinates of the nucleons are complex quantities. Effects of radiation damping appear as a natural consequence. The paper deals with the phenomenon of β-disintegration and meson decay. The meson field is considered to consist of a mixture of scalar and pseudo-scalar fields. The life of the pseudo-scalar meson becomes very short, 10^{-15} sec., while the life of the scalar meson can be adjusted by suitable choice of parameters to correspond to the observed lifetime of the mesons in cosmic rays. The question of the magnetic moments of the nucleons is also considered.

L. Jánossy (Manchester).

Yamasaki, Zyunpei. On the meson theory involving a mixture of scalar and pseudoscalar fields. II. Proc. Phys.-Math. Soc. Japan (3) **26**, 32–40 (1944). [MF 15081]

As an application of the theory developed in the paper reviewed above, the deuteron is considered theoretically. The energies of singlet and triplet states and the value of the quadrupole moment of the deuteron can be determined in agreement with the observational values, provided the constants are adjusted in a suitable manner.

L. Jánossy (Manchester).

Mazet, Robert. Sur une méthode permettant de trouver rapidement des formules de quantification utilisables en pratique (méthode dite des résidus). J. Math. Pures Appl. (9) **23**, 305–330 (1944). [MF 15649]

The author describes and illustrates by examples a simple method of finding quantization formulas, which is applicable to many of the problems of quantum mechanics. He assumes a differential equation of the form (1) $d^2\psi/dz^2+H(z)\psi=0$, where $H(z)$ is analytic except for isolated singularities and involves a parameter E which is to be quantized. The change of variables $W=\psi'/\psi$ reduces (1) to the Riccati equation (2) $W'+W^2+H(z)=0$. The singularities of $W(z)$ are either singularities of $H(z)$ or zeros of $\psi(z)$. The theory of residues is then used to obtain a formula of the type $E=E(q)$, where q is the number of zeros of $\psi(z)$. This is the desired quantization formula.

This method is then applied to the linear harmonic oscillator, the hydrogen atom and other familiar problems, and the well-known quantization formulas are quickly derived. The Stark effect for the hydrogen atom, as well as perturbation methods in general, are also treated. There are two appendices. In the first, the author's method is compared with the method of Brillouin, Wentzel and Kramers. The second appendix deals with the asymmetric top, with and without an external electric field, as in the usual treatment of rotational states of polyatomic molecules.

The peculiarity of the author's method is that it gives no information concerning the eigenfunctions, although leading rapidly to formulas for eigenvalues.

O. Frink.

Davidson, P. M. On a type of canonical transformation. Proc. Edinburgh Math. Soc. (2) **7**, 134–137 (1946). [MF 16208]

A formal quantum-theoretical analogue of the well-known determination of all contact transformations in classical dynamics.

D. C. Lewis (Durham, N. H.).

Harish-Chandra. A note on the σ-symbols. Proc. Indian Acad. Sci., Sect. A. **23**, 152–163 (1946). [MF 16274]

The symbols $\sigma^k_{\lambda\dot{\mu}}$ used to set up a connection between tensors and spinors for Lorentz transformations are discussed. Rather than give the symbols explicit numerical values, the author defines them by the conditions

$$(1)\qquad \overline{\sigma^k_{\mu\dot{\lambda}}}=\sigma^k_{\lambda\dot{\mu}},$$

$$(2)\qquad \sigma_k{}^{\lambda\dot{\nu}}\sigma^{k}_{\mu\dot{\rho}}=\delta_\rho{}^\lambda g_{\mu l}-(i/2)\epsilon_{klmn}\sigma^{m\lambda\dot{\nu}}\sigma^n_{\mu\dot{\rho}},$$

and points out that many well-known properties of the σ symbols are preserved. If σ and σ' are two different sets of symbols satisfying (1) and (2), then $\sigma'^k_{\lambda\dot{\mu}}=a_l{}^k\sigma^l_{\lambda\dot{\mu}}$, where the $a_l{}^k$ are the coefficients of a Lorentz transformation. Finally, the Dirac equation for a particle of spin $\frac{1}{2}$ is discussed in what is said to be a new way. One spinor and its derivatives are used to describe the particle instead of two spinors.

A. Schwartz (State College, Pa.).

Bhabha, H. J., and Harish-Chandra. On the theory of point-particles. Proc. Roy. Soc. London. Ser. A. **183**, 134–141 (1944). [MF 15822]

In Dirac's classical theory of radiating electrons [same Proc. **167**, 148–169 (1938)] it is shown that the singular terms of the flow of energy and momentum are perfect differentials along the world line of an electron and that equations of motion for an electron can be obtained by postulating that the nonsingular terms too are perfect differentials. In this paper it is shown that the same property

holds for a most general type of point-particle, represented by a superposition of arbitrary multipoles. The author's argument makes use of the conservation laws alone and not of any special features of a particular point-particle. It is also shown that the same argument applies to angular momentum. For the electron, it has been shown by Pryce [same Proc. 168, 389–401 (1938)] that, without altering the equations of motion, the energy-momentum tensor may be modified so as to make the flow of energy and momentum round the world line nonsingular. The authors show that this feature too holds for a general point-particle.

P. Weiss (London).

Harish-Chandra. On the removal of the infinite self-energies of point-particles. Proc. Roy. Soc. London. Ser. A. 183, 142–167 (1944). [MF 15823]

The energy-momentum tensor for a general point-particle was set up by Bhabha and Harish-Chandra in the paper reviewed above. It is shown how, by a step-by-step method, its singularities may be removed until the energy-momentum flow of the modified tensor is finite on the world-line of the particle. Special features of the electro-magnetic and the meson field are discussed. *P. Weiss* (London).

Bhabha, H. J., and Harish-Chandra. On the fields and equations of motion of point particles. Proc. Roy. Soc. London. Ser. A. 185, 250–268 (1946). [MF 15820]

Continuing the two papers reviewed above, the authors further extend Dirac's classical theory of radiating electrons to a general point-particle by treating the interaction of such a point-particle with a generalized wave field U. Here U stands for the component of a tensor of arbitrary rank and satisfies the generalized wave-equation $(g^{\mu\nu}\partial_\mu\partial_\nu+\chi)U=0$, where $g_{\mu\nu}$ are constants and χ is a constant scalar.

Let x^μ denote the coordinates of an arbitrary point in space-time, and let $z^\mu(\tau)$ denote the world-line of the particle. Then, instead of working in terms of a retarded or advanced point, defined by the equation $g_{\mu\nu}(x^\mu-z^\mu(\tau))(x^\nu-z^\nu(\tau))=0$, as previous writers have done, the authors define a "contemporary" point $z^\mu(\tau_c)$ by the equation $g_{\mu\nu}(x^\mu-z^\mu(\tau_c))v^\mu(\tau_c)=0$, $v^\mu=dz^\mu/d\tau$, and expand an arbitrary field quantity in terms of the displacement $l^\mu=x^\mu-z^\mu(\tau_c)$ from the contemporary point. The length of the perpendicular dropped from (x^μ) onto the world-line is denoted by ϵ and defined by $\epsilon^2=-g_{\mu\nu}l^\mu l^\nu$.

Following Dirac, the authors define a "radiation" field as the difference between the retarded and advanced field and a "symmetric" field which, for $\chi=0$, is half the sum of the retarded and advanced field and, for $\chi\neq0$, differs from this expression by an integral along the world-line from the retarded to the advanced point. This modification does not alter the singularities of the symmetric field. Then a "mean" field is defined by putting the actual field equal to the symmetric field plus the mean field and it is shown that all the singularities of the actual field are contained in the symmetric field which, if expanded in powers of ϵ, contains only odd powers of ϵ, beginning with the (negative) power corresponding to the order of the highest multipole of the particle. The same is true of all the derivatives of the symmetric field. On the other hand, the radiation field and all its derivatives are finite on the world-line.

The energy-momentum tensor, being quadratic in the actual field, is the sum of three terms, two of which are quadratic in the symmetric and mean fields, respectively, while the third is bilinear in those two fields. It is shown that the equations of motion of the particle depend only on the bilinear term, and it is, therefore, possible to remove the singularities from the energy-momentum flow by subtracting from the energy-momentum tensor the term which is quadratic in the symmetric field, without altering the equations of motion. This procedure differs from that which the authors, following Pryce, have adopted in their previous papers. *P. Weiss* (London).

Harish-Chandra. On the equations of motion of point particles. Proc. Roy. Soc. London. Ser. A. 185, 269–287 (1946). [MF 15821]

In the paper reviewed above it has been shown that the equations of motion of a general particle depend only on that part of the energy-momentum tensor which is bilinear in the "symmetric" and "mean" field. In this paper it is shown how the explicit form of the equations of motion can be obtained from a few general principles without the necessity of calculating all the terms of the bilinear part of the energy-momentum flow. The method reduces to the computation of the "radiation" field alone.

A definition of "spin angular momentum" of a general point-particle is given and it is postulated that the magnitude of the spin is constant. This postulate restricts the arbitrariness in the character of the multipoles. It is used in the actual determination of the equations of motion in the case of vector and scalar-meson fields. *P. Weiss.*

Bhabha, H. J. Relativistic wave equations for the proton. Proc. Indian Acad. Sci., Sect. A. 21, 241–264 (1945). [MF 15368]

The investigation of relativistically invariant wave equations of the form [cf. Rev. Modern Phys. 17, 200–216 (1945); these Rev. 7, 272] (1) $(\alpha^k p_k+\chi)\psi=0$ without any further auxiliary conditions is continued. It has been shown before that the matrices α^k can be infinitesimal operators of an irreducible representation of the five-dimensional Lorentz group. Four axes of this Lorentz group are the ordinary space-time axes, while the fifth has no physical significance. The α^k is the infinitesimal operator which corresponds to the rotation of the k axis toward the fifth axis. Since the irreducible representations of the Lorentz group can be characterized by two numbers, n and n', both integers or both half integers, the equations of the general form (1) can also be characterized by two such numbers. Since the α^k form an irreducible matrix system, the equation (1) is called irreducible by the author, although except for the equations corresponding to a few pairs n, n', the total linear manifold of the ψ satisfying (1) can be decomposed into linear submanifolds which are still invariant under Lorentz transformations. Thus, for example, it is possible to form submanifolds in which the mass $\sum p_k p^k$ is constant, while this quantity can assume several values for the total manifold defined by (1).

The present paper investigates the physical properties of a particle described by the ψ of (1). The effect of an electromagnetic field is introduced by replacing p_k by p_k-eA_k, where the A_k are the electromagnetic potentials. These added terms will, in general, induce transitions between the submanifolds mentioned above and also cause, in particular, transitions between the several possible mass values.

The author defines a Lagrange function from which (1) can be derived and gives expressions for the current density, the stress-energy tensor and the tensor for the angular

momentum density. The first two quantities can be defined in a way analogous to that used for Dirac's electron equation, with which the whole formalism shows a close similarity.

It is pointed out that the equations with half integer n, n' describe particles with half integer spin and obeying the exclusion principle; those with integer n, n', particles with integer spin and Bose statistics. It is also pointed out that the equation $n=\frac{3}{2}$, $n'=\frac{1}{2}$ may be the proper equation for the proton but no real arguments in favor of this assumption could be adduced at the present stage of the theory. In particular, the theory fails to give the observed magnitude of the magnetic moment of the proton. If the equation with $n=\frac{3}{2}$, $n'=\frac{1}{2}$ is the one which corresponds to the proton (and not $n=n'=\frac{1}{2}$ as commonly assumed), the proton should manifest, at high energies, an anomalous behavior, which can be derived from (1). *E. P. Wigner.*

Lewis, T. Some criticisms of the theory of point electrons. Philos. Mag. (7) **36**, 533–541 (1945). [MF 15950]

The author raises objections to Dirac's classical theory of point electrons and their interaction with electromagnetic fields [Communications Dublin Inst. Advanced Studies, Ser. A. no. 1 (1943); these Rev. **7**, 100]. The author claims that Dirac has committed a mathematical error in calculating the derivatives of combinations of the various electromagnetic potentials. This claim is the main basis for the various objections raised. Specifically, the author claims that one half the difference of the advanced and retarded potentials due to a moving point change at a point X^μ is not a function of X^μ alone but involves in addition functions of the proper time of the particle. Since the definition of retarded and advanced potentials determines the proper time used in these expressions as functions of the X^μ, the reviewer does not agree that an error has been committed. *A. H. Taub* (Seattle, Wash.).

Bopp, Fritz. Die Massenstabilität des Elektrons. Z. Naturforschung **1**, 53–58 (1946).

A study is made of the requirements which a theory of the electron must satisfy for which the field energy is finite and is considered to account for the entire electronic mass; five conditions are given. They are: (1) Coulomb condition: the static potential of a point charge gives the Coulomb field at large distances; (2) convergence condition: the field energy of a stationary point charge is finite and is determined by the mass; (3) photon condition: there are no other quanta besides the photons; (4) radiation condition: the force of radiation reaction coincides with that of Maxwell's theory; (5) mass stability condition: after interaction with an external field, the mass returns to its original equilibrium value. The first four conditions are not new but the last has not been considered heretofore. These conditions are formulated in a unified manner as mathematical conditions which the Laplace transform of the electrostatic potential must satisfy. Maxwell's theory, the author's meson theory of the electron and Dirac's theory of the electron are examined and shown to violate at least one of these conditions. One satisfactory solution is offered but the electrostatic potential turns out to be a very complicated function of the radius. *S. Kusaka.*

Cheng, Kai-Chia. Derivation of Dirac's equation for a free particle. Proc. Cambridge Philos. Soc. **42**, 185–187 (1946). [MF 15670]

Ville, J. Sur la relation d'incertitude de Heisenberg. Ann. Univ. Lyon. Sect. A. (3) **8**, 49–52 (1945). [MF 16304]

Nikolsky, K. V. On the formulation of the Heisenberg-Bohr indeterminacy principle for the quantum problem of interaction between elementary particles. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 24–27 (1945). [MF 16398]

Petiau, Gérard. Les systèmes de matrices de la représentation des corpuscules de spin $h/2\pi$. Revue Sci. **83**, 67–74 (1945). [MF 16856]

The matrices which appear in the linear wave equation for particles of spin 1 satisfy the commutation relations

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\delta_{\mu\nu}\beta_\rho+\delta_{\nu\rho}\beta_\mu$$

which were first obtained by Duffin. In the actual four-dimensional space there are four independent β's, but Kemmer [Proc. Cambridge Philos. Soc. **39**, 189–196 (1943); these Rev. **5**, 225] investigated the general case of n β's and obtained the number and rank of all the irreducible representations. In this paper the simple cases of $n=2, 3, 4$ and 5 are studied in detail and explicit representations are obtained. *S. Kusaka* (Princeton, N. J.).

Petiau, Gérard. Sur les relations entre densités de valeurs moyennes dans la théorie de l'électron relativiste de Dirac. Revue Sci. **83**, 37–39 (1945). [MF 16375]

Petiau, Gérard. Sur les relations entre densités de valeurs moyennes déduites de l'équation d'ondes de l'électron de Dirac. C. R. Acad. Sci. Paris **222**, 640–642 (1946). [MF 16046]

Viard, Jeannine. Emploi des intégrales premières en mécanique ondulatoire des systèmes. Propriétés du moment cinétique d'un système de corpuscules. C. R. Acad. Sci. Paris **222**, 53–55 (1946). [MF 15979]

van Isacker, Jacques. Quelques formules relatives aux ondes planes de corpuscules de spin $nh/4\pi$. C. R. Acad. Sci. Paris **222**, 375–376 (1946). [MF 16016]

Slansky, Serge. Sur le centre de gravité de deux corpuscules de spin 1/2. C. R. Acad. Sci. Paris **222**, 994–996 (1946). [MF 16392]

Potier, Robert. Sur la définition et les propriétés du vecteur-courant associé à un corpuscule de spin quelconque. C. R. Acad. Sci. Paris **222**, 1076–1079 (1946). [MF 16524]

Mosharrafa, A. M. Wave surfaces associated with world-lines. Proc. Math. Phys. Soc. Egypt **2**, no. 2, 63–66 (1944). [MF 16340]

The phase and group velocities of the de Broglie wave associated with uniformly moving particles are interpreted geometrically by means of wave surfaces. *A. E. Schild.*

Hylleraas, Egil A. On the solution of the relativistic wave equation for an electron in a Coulomb field. Avh. Norske Vid. Akad. Oslo. I. **1943**, no. 1, 15 pp. (1943). [MF 16435]

This paper begins with a lengthy review of the usual method for writing the Dirac equation for an electron in a spherically symmetric Coulomb field. The wave functions are written as products of functions depending on the distance from the center of the force field, the radial functions,

and functions depending on the angles. In the usual treatment of this problem, two radial functions are introduced and are determined by solving a pair of first order differential equations. The author shows that both of these functions may be expressed in terms of a single function and its first derivative. This auxiliary function satisfies a second order differential equation and is expressible as a confluent hypergeometric series. *A. H. Taub.*

Blokhinzev, D. On the theory of the motion of a particle in the Coulomb field. Acad. Sci. USSR. J. Phys. **10**, 196 (1946).

Broch, Einar Klaumann. On the theory of the fine structure of the hydrogen atom. Arch. Math. Naturvid. **46**, no. 3, 47–125 (1943). [MF 15806]

This is a careful and detailed study of the hydrogen atom problem both on the basis of the nonrelativistic quantum theory of Schrödinger and of the relativistic quantum mechanics of Dirac. Instead of assuming that the nucleus is a point charge, which is the center of a Coulomb force of attraction down to $r=0$, where r is the separation of nucleus and electron, the author takes into account the finite size of the nucleus by assuming a Coulomb attraction down to $r=r_0$, which changes into a repulsion for smaller values of r. Here r_0, the radius of the proton, is taken to be of the same order of magnitude as the classical electron radius.

Applying perturbation methods, and with various assumptions concerning the law of repulsion for $r<r_0$, including both weak and strong repulsions, the author finds upper and lower limits to the possible shift in energy levels for different states of the atom due to this effect. These possible energy shifts are of the right order of magnitude to explain certain slight discrepancies between experimental values and the fine structure formula of Sommerfeld, obtained from the point-charge model.

In applying perturbation methods in the nonrelativistic case, the author's unperturbed state is more complicated than Schrödinger's. Since boundary conditions for the radial equation are given for $r=r_0$ instead of $r=0$, the usual associated Laguerre polynomials are replaced by Whittaker's confluent hypergeometric functions. The mathematical treatment of the Dirac theory is more complicated still, but the numerical results obtained are about the same for the two theories. More positive corroboration of the author's theory would require further experimental work on the fine structure of the hydrogen spectrum. *O. Frink.*

Thermodynamics, Statistical Mechanics

*Schrödinger, Erwin. **Statistical Thermodynamics.** Cambridge, at the University Press; New York, The Macmillan Company, 1946. vi+88 pp. $1.50.

The author seeks to present "one simple, unified standard method" for dealing with the various problems of statistical thermodynamics. To focus attention on the scope and validity of this general procedure, many conventional topics are severely condensed, while vital points usually passed over are dealt with at greater length.

The text begins with the statement that "there is, essentially, only one problem in statistical thermodynamics: to determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E." Chapter I elaborates upon this statement; in particular, the statistical interpretations of Maxwell and Gibbs are outlined and compared. In chapter II the number of possible states of an assembly is formulated, the most probable distribution is determined with the aid of Lagrange multipliers λ, μ, the physical meaning of μ is worked out in detail and the statistical meaning of free energy is given. Chapter III gives a brief but careful interpretation of the Nernst theorem. A free mass-point, a Planck oscillator, and a Fermi oscillator are introduced in chapter IV. A critique is given in chapter V of a common proof that the deviations from the most probable distribution can be neglected. An alternative proof, by the method of mean values, is presented in chapter VI, where the use of residues is given in careful detail. Several statistical definitions of entropy are developed. Chapter VII applies the method of residues to develop the thermodynamics of "gases," a "gas" being composed of either photons, atoms or electrons. The results obtained are applied in chapter VIII to the determination of the entropy constant, to an explanation of Gibbs's paradox, and to a study of various types of "gas" degeneration. Chapter IX develops the theory of radiation from the point of view of Peng and Born.

An outstanding feature of this book is the author's effective combination of heuristic reasoning and rigor, coupled with candor. Even though the author says that this book is "not a first introduction" to the subject, it nevertheless serves admirably as such an introduction.

C. C. Torrance (Annapolis, Md.).

Popoff, Kyrille. Alcune osservazioni di natura matematica sui due principi della termodinamica. Boll. Un. Mat. Ital. (2) **5**, 141–144 (1943). [MF 16099]

The author derives the integrating factor $1/T$ for entropy by elementary methods without reference to the second law of thermodynamics. *C. C. Torrance* (Annapolis, Md.).

Schouls, Georgette. Computation of radiation pressure by application of statistical mechanics. Wis- en Natuurk. Tijdschr. **10**, 155–161 (1941). (Dutch. French summary) [MF 15579]

Le but de ce travail est de montrer comment la mécanique statistique générale permet d'obtenir d'une manière très simple la pression de radiation s'exerçant sur les faces intérieures d'un corps solide isotrope par suite de l'agitation thermique. *From the author's summary.*

Takahasi, Hidetosi. Eine einfache Methode zur Behandlung der statistischen Mechanik eindimensionaler Substanzen. Proc. Phys.-Math. Soc. Japan (3) **24**, 60–62 (1942). [MF 15019]

The partition function for a linear gas consisting of $n+1$ molecules moving on a line, with attraction only between next neighbors, is, without any approximation, reduced to the nth power of an integral for a pair. As a consequence the coexistence of two phases is found impossible for this model, in contradiction with Harasima [Proc. Phys.-Math. Soc. Japan (3) **21**, 156–165 (1939)]. *F. Zernike.*

DECEMBER ISSUE IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).